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SPECIAL REPRESENTATIONS OF RANK ONE ORTHOGONAL GROUPS

BY

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ABSTRACT

We study the image of the theta correspondence from $\overline{SL}(2)$ to a rank one orthogonal group (over a number field). The image consists of cusp forms, the Fourier coefficients of which satisfy a certain invariance property. We show that this property characterizes the image. The proof requires first an analogous local statement (almost everywhere) and then a use of certain Rankin–Selberg integrals.

Introduction

Let k be a number field and X a vector space over k equipped with a nondegenerate symmetric bilinear form $(\ ,\ )$ defined over k, with Witt index one. Assume that $m = \dim X \geqq 5$, and put $G = SO(X)$, the special orthogonal group of $(X, (\ ,\ ))$. In this paper we characterize the image of the theta correspondence from $\overline{SL}(2, \mathbb{A})$ (the metaplectic group in case m is odd, and $SL(2, \mathbb{A})$ in case m is even), to $G(\mathbb{A})$. ($\mathbb{A}$ is the ring of adeles of k.) The characterization is in terms of a certain property of the Fourier coefficients of cusp forms on $G(\mathbb{A})$. We call such cusp forms *special*. To explain this term, we introduce some notation. Note that G is a group of rank one over k. Write

$$X_k = ke_1 + L_k + ke_{-1} \tag{1}$$

where e_1, e_{-1} are isotropic vectors, $(e_1, e_{-1}) = 1$ and L_k is the orthogonal complement of $ke_1 + ke_{-1}$. We write the elements of G according to (1).

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Denote by P the parabolic subgroup of G, which preserves ke_1. (We assume a left action of G on X.) The elements of the unipotent radical N_k have the form

$$\eta(v)=\begin{pmatrix}1 & v^* & -\frac{1}{2}(v,v)\\ & I_L & v\\ & & 1\end{pmatrix}$$

where $v\in\mathrm{Hom}(ke_{-1},L_k)=L_k$ and v^* is the linear functional on L_k which sends $l\in L_k$ to $-(v,l)$.

Let φ be a cusp form on $G(\mathbb{A})$, then it has a Fourier expansion

$$\varphi(h)=\sum_{0\neq l\in L_k}\varphi_l(h)$$

where

$$\varphi_l(h)=\int_{L_k\backslash L_{\mathbb{A}}}\psi^{-1}((v,l))\varphi(\eta(v)h)dv,$$

ψ is a fixed nontrivial character of $k\backslash\mathbb{A}$. Denote for $0\neq l\in L_k$ by O_l the stabilizer of $\eta(l)$ in M, the Levi part of P. The elements of O_l have the form

$$\begin{pmatrix}\varepsilon & & \\ & \tilde{h} & \\ & & \varepsilon^{-1}\end{pmatrix}$$

where $\varepsilon=\pm 1$ and $\tilde{h}\in\mathrm{SO}(L)$ satisfies $\tilde{h}\cdot l=\varepsilon l$. (Note that $(l,l)\neq 0$, and hence O_l is the orthogonal group of the nondegenerate space $(l)^{\perp}\cap L$.) We clearly have

$$\varphi_l(\delta h)=\varphi_l(h),\qquad\forall\,\delta\in O_l(k).$$

Denote by O_l^c the connected component of O_l. We say that the cusp form $\varphi(h)$ is *special* if for any $0\neq l\in L_k$

$$\varphi_l(\delta h)=\varphi_l(h),\qquad\forall\,\delta\in O_l^c(\mathbb{A}).$$

An irreducible, automorphic, cuspidal representation π of $G(\mathbb{A})$ is called *special* if all its cusp forms are special. It is straightforward to see that a representation π which lies in the image of the theta correspondence from $\overline{\mathrm{SL}}(2,\mathbb{A})$ is special. This is shown in Section 1. The main theorem of this paper is that the converse holds. For that we introduce local analogues of special representations. Such representations satisfy a certain "U-property": Let π_v be an irreducible admissible representation of $G_v = G(k_v)$, we say that π_v satisfies the U-property if for every nonisotropic $l \in L_v$ and any linear functional γ on V_{π_v} (the space of π_v), satisfying

$$\gamma(\pi(\eta(u))\zeta) = \psi((u, l))\gamma(\zeta), \qquad \forall \zeta \in V_{\pi_v}, \quad \forall u \in L_v$$

γ also satisfies

$$\gamma(\pi(\delta)\zeta) = \gamma(\zeta) \qquad \forall \delta \in O_l^c$$

(O_l^c is defined similarly over k_v). In Sections 2 and 5 we prove the analogous local theorem for unitary representations with U-property. We prove this under the assumption that v is finite and that the rank of G_v is at least two. Note that since $\dim L \geqq 3$, then G_v is of rank at least two for almost all places v. In Section 2 we prove that a representation of G_v, obtained by the local theta correspondence from $\overline{\mathrm{SL}}(2, k_v)$, satisfies the U-property. The proof is a local analogue of the proof given in Section 1. In Section 5, after some preparation, we prove the converse. We show that an irreducible unitary representation with U-property restricts on a certain subgroup Q to an irreducible representation (Theorem 5.3). Next we prove (Theorem 5.4) that two irreducible unitary representations with U-property, having the same restriction on Q, are isomorphic. This together with an explicit description of the local theta correspondence from $\overline{\mathrm{SL}}(2, k_v)$ [S], implies the (local) theorem.

To obtain the global theorem, we introduce certain Rankin–Selberg integrals

$$I_H(s) = \int_{H(k)\backslash H(\mathbf{A})} \varphi(h)E(f_s, h)dh, \qquad \varphi \in \pi.$$

H is the stabilizer in G of a nonzero element of L_k. $E(f_s, h)$ is a certain special Eisenstein series on H. It turns out that $I_H(s)$ has a (simple) pole at $s = (m-3)/2$ (for some H) if and only if π lifts to $\overline{\mathrm{SL}}(2, \mathbf{A})$. On the other hand the integral $I_H(s)$ is (essentially) Eulerian, and can be computed explicitly, since for almost all v, π_v has the U-property and hence is obtained (by the theta-correspondence) from a (unitary) representation of $\overline{\mathrm{SL}}(2, k_v)$. The computation shows that $I_H(s)$ has a pole at $s = (m-3)/2$ (for appropriate H). We remark that $I_H(s)$ yields the standard L-function for π, provided $E(f_s, h)$ is properly normalized. However, $I_H(s)$ generally vanishes identically except for a small class of cuspidal representations (which include special representations).

It is not hard to see that with slight modifications our proof gives the similarly theorem for the case where the Witt index of X is two. (Here a cusp form φ is *special* if φ_l is $O_l^c(\mathbf{A})$ left-invariant, whenever $l \in L_k$ is nonisotropic.) This case was studied by Piatetski-Shapiro [PS] for the group $\mathrm{PGSp}(4) = \mathrm{SO}(3, 2)$. Our terminology is borrowed from this work. The local theorems presented here generalize those in [PS]. The idea is to focus attention on the subgroup Q, and then the proofs become essentially the same. We present the proofs in a slightly different style, adapted to Mackey theory. The global integral $I_H(s)$ was introduced in order to overcome the fact that G may remain a rank one group in a finite number of places.

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§1. Theta-lifts from $\overline{\mathrm{SL}}(2, \mathbf{A})$ to $G(\mathbf{A})$ are special

Let σ be an irreducible, automorphic, cuspidal representation of $\overline{\mathrm{SL}}(2, \mathbf{A})$. Fix a nontrivial character ψ of $k \setminus \mathbf{A}$, and denote by $\theta(\sigma, \psi)$ the theta-lift of σ to $G(\mathbf{A})$. In this section we show that $\theta(\sigma, \psi)$ is special. We employ the notation already established in the introduction. Let Y be the two dimensional symplectic space defined over k, with symplectic form $\langle\ ,\ \rangle$. We let SL(2) act on Y from the right. The reductive dual pair $(\mathrm{SL}(2), G)$ is embedded inside $\mathrm{Sp}(Z)$, $Z = Y \otimes X$ (with symplectic form $\langle\ ,\ \rangle \otimes (\ ,\ )$). Let ω_ψ be the Weil representation of $\widetilde{\mathrm{Sp}}(Z_{\mathbf{A}})$ realized in $L^2(Z_{\mathbf{A}}^-)$ where $Z = Z^+ + Z^-$ is a given polarization of Z. The space of $\theta(\sigma, \psi)$ is generated by the forms

$$\varphi(h) = \int_{\mathrm{SL}(2,k)\setminus \mathrm{SL}(2,\mathbf{A})} \theta_\psi^\phi(g, h)\xi(g)dg, \qquad h \in G(\mathbf{A}) \tag{1.1}$$

where $\xi \in \sigma$ and $\theta_\psi^\phi(g, h)$ is the theta-series

$$\theta_\psi^\phi(g, h) = \sum_{z \in Z_k^-} \omega_\psi(g, h)\phi(z), \qquad \phi \in S(Z_{\mathbf{A}}^-).$$

In this section we choose the following polarization of Z. Let $\{\varepsilon_+, \varepsilon_-\}$ be a basis of Y_k, such that $\varepsilon_\pm$ are isotropic and $\langle \varepsilon_+, \varepsilon_- \rangle = 1$, (we represent SL(2) according to this basis), then consider the polarization

$$Z = Z^+ + Z^-,$$

$$Z^\pm = Y \otimes e_{\pm 1} + \varepsilon_\pm \otimes L.$$

We identify $Z^- = Y \oplus L$. We will need the following formulas. (For formulas for ω_ψ and generalities see [R] for example.)

(1.2) For $\phi \in S(Y_{\mathbb{A}} \oplus L_{\mathbb{A}})$, $v \in L_{\mathbb{A}}$

$$\omega_\psi(1, \eta(v))\phi(y; l) = \psi(\tfrac{1}{2}(2(v, l)y^+ - y^+y^-(v, v)))\phi(y; l - y^-v)$$

where $y = y^+\varepsilon_+ + y^-\varepsilon_-$

(1.3) $$\omega_\psi\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, 1\right)\phi(y; l) = \psi(-\tfrac{1}{2}x(l, l))\phi\left(y \cdot \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}; l\right).$$

Put

$$U = \left\{u(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right\}.$$

Proposition 1.1. *Assume that σ does not lift (via $\theta(\cdot, \psi)$) to* $\mathrm{SO}(L_{\mathbb{A}})$, *then $\theta(\sigma, \psi)$ is nonzero if and only if σ has a Whittaker model with respect to U and a character*

$$\psi^{\lambda/2}(u(x)) = \psi\left(\frac{\lambda}{2}x\right)$$

where $\lambda \in k^*$ is represented by the quadratic form on L_k, i.e., there is $l \in L_k$ such that $(l, l) = \lambda$.

Proof. $\theta(\sigma, \psi)$ is nonzero if and only if there is $0 \neq l_0 \in L_k$ such that $\varphi_{l_0}(1) \not\equiv 0$ for φ of the form (1.1). (By our assumption on σ, $\varphi_0(1) \equiv 0$.) Let us first show that

$$\varphi(h) = \int_{U_k \backslash \mathrm{SL}_2(\mathbb{A})} \sum_{l \in L_k} \omega_\psi(g, h)\phi(\varepsilon_+; l)\xi(g)dg.$$

For this, we may assume that $h = 1$ and $\phi = \phi_1 \otimes \phi_2$, where $\phi_1 \in S(Y_{\mathbb{A}})$ and $\phi_2 \in S(L_{\mathbb{A}})$. Then it follows that

$$\theta_\psi^\phi(g, 1) = \left(\sum_{y \in Y_k} \phi_1(yg)\right)\tilde{\theta}_\psi^{\phi_2}(g, 1), \qquad g \in \overline{\mathrm{SL}}(2, \mathbb{A})$$

where $\tilde{\theta}_\psi^{\phi_2}(g, 1)$ is the theta-series corresponding to the dual pair $(\mathrm{SL}(2), \mathrm{SO}(L))$. Thus

$$\theta_\psi^\phi(g, 1) = \phi_1(0)\tilde{\theta}_\psi^{\phi_2}(g, 1) + \sum_{\gamma \in U_k \backslash \mathrm{SL}(2,k)} \phi_1(\varepsilon_+ \gamma g)\tilde{\theta}_\psi^{\phi_2}(g, 1)$$

and hence

$$\varphi(1)=\phi_1(0)\int_{SL(2,k)\backslash SL(2,A)}\tilde{\theta}_w^{\phi_2}(g,1)\xi(g)dg+\int_{U_k\backslash SL(2,A)}\phi_1(\varepsilon_+g)\tilde{\theta}_\psi^{\phi_2}(g,1)\xi(g)dg$$
$$=\int_{U_k\backslash SL(2,A)}\phi_1(\varepsilon_+g)\tilde{\theta}_\psi^{\phi_2}(g,1)\xi(g)$$
$$=\int_{U_k\backslash SL(2,A)}\sum_{l\in L_k}\omega_\psi(g,1)\phi(\varepsilon_+;l)\xi(g)dg.$$

We used our assumption on σ. Let $l_0\in L_k$ be nonzero. We will obtain a simple formula for φ_{l_0}. Using (1.2)

$$\varphi_{l_0}(1)=\int_{L_k\backslash L_A}\psi^{-1}((v,l_0))\int_{U_k\backslash SL(2,A)}\sum_{l\in L_k}\omega_\psi(g,\eta(v))\phi(\varepsilon_+;l)\xi(g)dg\,dv$$
$$=\int_{U_k\backslash SL(2,A)}\int_{L_k\backslash L_A}\psi^{-1}((v,l_0))\sum_{l\in L_k}\psi((v,l))\omega_\psi(g,1)\phi(\varepsilon_+;l)\xi(g)dv\,dg$$
$$=\int_{U_k\backslash SL(2,A)}\omega_\psi(g,1)\phi(\varepsilon_+;l_0)\xi(g)dg.$$

(We assume that the volume of $L_k\backslash L_A$ is one.) Using (1.3) we get

$$\int_{U_A\backslash SL(2,A)}\omega_\psi(g,1)\phi(\varepsilon_+;l_0)\int_{k\backslash A}\psi^{-1}(\tfrac{1}{2}(l_0,l_0)x)\xi(u(x)g)dx\,dg$$

Thus

$$\varphi_{l_0}(1)=\int_{U_A\backslash SL(2,A)}\omega_\psi(g,1)\phi(\varepsilon_+;l_0)w_{\xi,(l_0,l_0)/2}(g)dg \tag{1.4}$$

where

$$w_{\xi,\lambda}(g)=\int_{k\backslash A}\psi^{-1}(\lambda x)\xi(u(x)g)dx.$$

Formula (1.4) shows that if σ has no Whittaker model with respect to U and the character $\psi^{(l_0,l_0)/2}$ for any $0\neq l_0\in L_k$ (in the sense that $w_{\xi,(l_0,l_0)/2}\equiv 0$), then $\theta(\sigma,\psi)\equiv 0$. Now assume that σ has a Whittaker model with respect to U and $\psi^{(l_0,l_0)/2}$, then $\varphi_{l_0}(1)\not\equiv 0$, since otherwise, for $\phi=\phi_1\otimes\phi_2$ as above we get from (1.4)

$$\int_{U_A\backslash SL(2,A)}\phi_1(\varepsilon_+g)\tilde{\omega}_\psi(g,1)\phi_2(l_0)w_{\xi,(l_0,l_0)/2}(g)dg\equiv 0,\qquad \forall\phi_1\in S(Y_A).$$

This implies that $\tilde{\omega}_\psi(g,1)\phi_2(l_0)w_{\xi,(l_0,l_0)/2}(g)\equiv 0$, which is impossible. ($\tilde{\omega}_\psi$ is the Weil representation corresponding to the dual pair (SL(2), O(L)).) The proposition is proved. □

Formula (1.4) implies

THEOREM 1.2. *Let σ be as in Proposition* 1.1, *then* (*if* $\theta(\sigma, \psi) \neq 0$), $\theta(\sigma, \psi)$ *is a special representation.*

PROOF. Let $l_0 \in L_k$ be nonzero, and let δ be in $O^c_{l_0}(\mathbf{A})$. δ has the form

$$\delta = \begin{pmatrix} 1 & & \\ & \tilde{\delta} & \\ & & 1 \end{pmatrix}, \qquad \tilde{\delta} \cdot l_0 = l_0.$$

It is easy to see that

$$\omega_\psi(1, \delta)\phi(\varepsilon_+; l_0) = \phi(\varepsilon_+; l_0).$$

Thus (1.4) implies that

$$\varphi_{l_0}(\delta h) = \varphi_{l_0}(h) \qquad \forall h \in G(\mathbf{A}), \quad \varphi \in \theta(\sigma, \psi). \qquad \square$$

§2. Local theta-lifts from $\overline{\mathrm{SL}}(2, F)$ to G satisfy the U-property

Let F be a local nonarchimedean field and X a vector space equipped with a nondegenerate symmetric bilinear form, having a positive Witt index. We assume that $m = \dim X \geqq 5$ and put $G = \mathrm{SO}(X)$. We write again $X = Fe_1 + L + Fe_{-1}$ as in (1) in the introduction, only that now L is not necessarily anisotropic. We will employ the local notation analogous to that already used in the previous sections (e.g., Y is the two dimensional symplectic space of F, ω_ψ is the Weil representation corresponding to the dual pair $(\overline{\mathrm{SL}}(2, F), G)$, etc.).

PROPOSITION 2.1. *Let σ be an irreducible, admissible representation of $\overline{\mathrm{SL}}(2, F)$, and π an irreducible, admissible representation of G such that $\mathrm{Hom}_{\overline{\mathrm{SL}}(2,F)\times G}(\omega_\psi, \hat{\sigma} \otimes \pi) \neq 0$. Let $l_0 \in L$ be nonisotropic, then the space of linear functionals γ on V_π (the space of π) satisfying*

$$(2.1) \qquad \gamma(\pi(\eta(v))\varphi) = \psi((v, l_0))\gamma(\varphi), \qquad v \in L, \quad \varphi \in V_\pi$$

is at most one dimensional.

PROOF. The proof runs in complete analogy to the proof of Proposition 1.1. We realize ω_ψ in $S(Y \oplus L)$ as in Section 1. (The analogues of (1.2), (1.3) are of course valid.) Let $\tilde{T}$ be a nonzero element in $\mathrm{Hom}_{\overline{\mathrm{SL}}(2,F)\times G}(\omega_\psi, \hat{\sigma} \otimes \pi) = \mathrm{Hom}_{\overline{\mathrm{SL}}(2,F)\times G}(\omega_\psi \otimes \sigma, \pi)$, and γ a nonzero linear functional, satisfying

(2.1). The composition $T = \gamma \circ \tilde{T}$ defines a bilinear form $T: S(Y \oplus L) \times V_\sigma \to \mathbf{C}$ (V_σ — the space of σ), satisfying

$$(2.2) \qquad T(\omega_\psi(g, 1)\phi, \sigma(g)\xi) = T(\phi, \xi), \qquad \phi \in S(Y \oplus L), \quad \xi \in V_\sigma,$$

$$(2.3) \qquad T(\omega_\psi(1, \eta(v))\phi, \xi) = \psi((v, l_0))T(\phi, \xi), \qquad v \in L.$$

We will show that the space of bilinear forms satisfying (2.2) and (2.3) is at most one dimensional. Put $\psi_{l_0}(\eta(v)) = \psi((v, l_0))$, then for ξ fixed, T defines a linear functional on the Jacquet module $S(Y \oplus L)_{N,\psi_{l_0}}$. Let $C = \{(0; l) \mid l \in L\}$ and $\mathcal{O} = \{(y; l) \mid 0 \neq y \in Y, l \in L\}$, then by [B.Z], we have the exact sequence

$$0 \to S(\mathcal{O}) \to S(Y \oplus L) \to S(C) \to 0.$$

Applying the Jacquet functor with respect to N and ψ_{l_0}, we get the exact sequence

$$0 \to S_{N,\psi_{l_0}}(\mathcal{O}) \to S_{N,\psi_{l_0}}(Y \oplus L) \to S_{N,\psi_{l_0}}(C) \to 0.$$

Since N acts trivially on $S(C)$, we have $S_{N,\psi_{l_0}}(C) = 0$. Thus we have an isomorphism

$$S_{N,\psi_{l_0}}(\mathcal{O}) \xrightarrow{\sim} S_{N,\psi_{l_0}}(Y \oplus L).$$

So we have to prove that the space of bilinear forms T on $S(\mathcal{O}) \times V_\sigma$ satisfying (2.2), (2.3) is at most one dimensional. Let $\phi \in S(\mathcal{O})$ be of the form $\phi_1 \otimes \phi_2$, where $\phi_1 \in S(Y \setminus \{0\})$ and $\phi_2 \in S(L)$. We have for $g \in \overline{\mathrm{SL}}(2, F)$

$$\omega_\psi(g, 1)\phi = \rho(g)\phi_1 \otimes \tilde{\omega}_\psi(g, 1)\phi_2$$

where $\rho(g)\phi_1(y) = \phi_1(yg)$ and $\tilde{\omega}_\psi$ is the Weil representation attached to the pair $(\mathrm{SL}(2), \mathrm{SO}(L))$. We then may think of T as a bilinear form T' on $S(Y \setminus \{0\}) \times (S(L) \otimes V_\sigma)$ satisfying

$$(2.2)' \qquad T'(\rho(g)\phi_1, \tilde{\omega}_\psi(g, 1)\phi_2 \otimes \sigma(g)\xi) = T'(\phi_1, \phi_2 \otimes \xi).$$

Fix ϕ_2 and ξ, then $T'(\cdot, \phi_2 \otimes \xi)$ is an $\mathrm{SL}(2, F)$-smooth distribution on $Y \setminus \{0\}$, and hence, fixing an $\mathrm{SL}(2, F)$-right invariant measure dy on $Y \setminus \{0\}$, there is a (unique) smooth function $f_{\phi_2 \otimes \xi}$ on $Y \setminus \{0\}$, such that

$$T'(\phi_1, \phi_2 \otimes \xi) = \int_{Y \setminus \{0\}} f_{\phi_2 \otimes \xi}(y)\phi_1(y)dy.$$

The condition (2.2)′ translates into

$$(2.4) \qquad f_{\tilde{\omega}_\psi(g,1)\phi_2 \otimes \sigma(g)\xi}(y) = f_{\phi_2 \otimes \xi}(yg), \qquad g \in \mathrm{SL}(2, F).$$

Realizing $Y\setminus\{0\} = U\setminus \mathrm{SL}(2,F)$ by $y = \varepsilon_+ g$, and replacing dy by an $\mathrm{SL}(2,F)$-right invariant measure on $U\setminus\mathrm{SL}(2,F)$, we get

$$T(\phi_1\otimes\phi_2,\xi) = T'(\phi_1,\phi_2\otimes\xi) = \int_{U\setminus \mathrm{SL}(2,F)} f_{\phi_2\otimes\xi}(\varepsilon_+ g)\phi_1(\varepsilon_+ g)dg$$
$$= \int_{U\setminus \mathrm{SL}(2,F)} f_{\tilde{\omega}_\psi(g,1)\phi_2\otimes\sigma(g)\xi}(\varepsilon_+)\phi_1(\varepsilon_+ g)dg.$$

Define the following bilinear form on $S(\mathcal{O})\times V_\sigma$. Let $\phi\in S(\mathcal{O})$ and $\xi\in V_\sigma$. Write

$$\phi = \sum_{i=1}^{d} \phi_1^{(i)}\otimes\phi_2^{(i)},$$

where $\phi_1^{(i)}\in S(Y\setminus\{0\})$ and $\phi_2^{(i)}\in S(L)$, then define

$$\tau(\phi,\xi) = \sum_{i=1}^{d} f_{\phi_2^{(i)}\otimes\xi}(\varepsilon_+)\phi_1^{(i)}(\varepsilon_+).$$

Rewriting

$$\tau(\phi,\xi) = f_{\sum_{i=1}^{d}\phi_1^{(i)}(\varepsilon_+)\phi_2^{(i)}\otimes\xi}(\varepsilon_+),$$

we see that τ is well defined and that $\tau(\phi,\xi)$ depends on ξ and on the restriction of ϕ to $X_0 = \{(\varepsilon_+, l) \mid l\in L\}$. Thus

$$T(\phi,\xi) = \int_{U\setminus \mathrm{SL}(2,F)} \tau(\omega_\psi(g,1)\phi,\sigma(g)\xi)dg.$$

By formula (1.2), we have

$$\mathrm{Res}_{X_0}\,\omega_\psi(g,\eta(v))\phi = \mathrm{Res}_{X_0}\,\phi'_{g,v}$$

where

$$\phi'_{g,v}(y;l) = \psi((v,l))\omega_\psi(g,1)\phi(y;l)$$

and hence

$$\tau(\omega_\psi(g,\eta(v))\phi,\sigma(g)\xi) = \tau(\phi'_{g,v},\sigma(g)\xi).$$

Thus, using (2.3)

$$T(\phi,\xi) = \psi^{-1}((v,l_0))T(\omega_\psi(1,\eta(v))\phi,\xi)$$
$$= \int_{U\setminus \mathrm{SL}(2,F)} \psi^{-1}((v,l_0))\tau(\phi'_{g,v},\sigma(g)\xi)dg$$
$$= \int_{U\setminus \mathrm{SL}(2,F)} \psi^{-1}((v,l_0))\,f_{\mathrm{Res}_{X_0}\phi'_{g,v}\otimes\sigma(g)\xi}(\varepsilon_+)dg$$
$$= \int_{U\setminus \mathrm{SL}(2,F)} \psi^{-1}((v,l_0))\,f_{\psi((v,\cdot))\tilde{\omega}_\psi(g,1)\phi_2\otimes\sigma(g)\xi}(\varepsilon_+)\phi_1(\varepsilon_+ g)dg.$$

This implies that

$$f_{\psi((v,\cdot))\tilde{\omega}_\psi(g,1)\phi_2\otimes\sigma(g)\xi}(\varepsilon_+) = \psi((v, l_0))\, f_{\tilde{\omega}_\psi(g,1)\phi_2\otimes\sigma(g)\xi}(\varepsilon_+)$$

and in particular

$$f_{\psi((v,\cdot))\phi_2\otimes\xi}(\varepsilon_+) = \psi((v, l_0))\, f_{\phi_2\otimes\xi}(\varepsilon_+).$$

This implies that the distribution $\phi_2 \mapsto f_{\phi_2\otimes\xi}(\varepsilon_+)$ is supported on $l = l_0$, and hence

$$f_{\phi_2\otimes\xi}(\varepsilon_+) = \alpha(\xi)\phi_2(l_0).$$

Since $\varepsilon_+ u = u$ for $u \in U$, we have

$$f_{\tilde{\omega}_\psi(u,1)\phi_2\otimes\sigma(u)\xi}(\varepsilon_+) = f_{\phi_2\otimes\xi}(\varepsilon_+)$$

and so

$$\alpha(\sigma(u)\xi)\tilde{\omega}_\psi(u, 1)\phi_2(l_0) = \alpha(\xi)\phi_2(l_0).$$

Since

$$\tilde{\omega}_\psi(u(x), 1)\phi_2(l_0) = \psi(-\tfrac{1}{2}(l_0, l_0)x)\phi_2(l_0),$$

we find that α is a Whittaker functional for σ, with respect to U and the character $\psi^{(l_0,l_0)/2}$. We proved

$$T(\phi, \xi) = \int_{U\backslash \overline{\mathrm{SL}}(2,F)} \omega_\psi(g, 1)\phi(\varepsilon_+; l_0)\alpha(\sigma(g)\xi)dg \tag{2.5}$$

and so T is uniquely determined up to a constant, and the proposition is proved. □

As in Section 1, formula (2.5) implies

Theorem 2.2. *Let π be an irreducible, admissible representation of G, lying in the image of the local theta-correspondence from $\overline{\mathrm{SL}}(2, F)$, then π satisfies the U-property.*

Proof. The assumption on π means that there is an irreducible, admissible representation σ of $\overline{\mathrm{SL}}(2, F)$, such that $\mathrm{Hom}_{\overline{\mathrm{SL}}(2,F)\times G}(\omega_\psi \otimes \sigma, \pi) \neq 0$. Let $\tilde{T}$ be a nonzero element of the last space, and let γ be a linear functional on V_π, satisfying (2.1) (for a given nonisotropic vector l_0 in L). Let $\varphi \in V_\pi$, and assume $\varphi = \tilde{T}(\phi, \xi)$, for $\phi \in S(Y \oplus L)$ and $\xi \in V_\sigma$, then for δ in $O^c_{l_0}$, we have

$$\gamma(\pi(\delta)v) = \gamma(\pi(\delta)\tilde{T}(\phi, \xi)) = \gamma(\tilde{T}(\omega_\psi(1, \delta)\phi, \xi))$$
$$= T(\omega_\psi(1, \delta)\phi, \xi) = T(\phi, \xi) = \gamma(v).$$

We used the notation of the last proof and formula (2.5). □

§3. Explicit description of the local theta-correspondence

In this section we summarize and recall from [S] the explicit construction of the local theta-correspondence from the dual of $\overline{\mathrm{SL}}(2, F)$ to G. We keep the notation of Section 2, and we add the assumption that the Witt index of the quadratic form on X is a least two (so that $(\mathrm{SL}(2, F), G)$ is a dual pair in the stable range).

Let σ be an irreducible, unitary representation of $\overline{\mathrm{SL}}(2, F)$ (assumed to be genuine in case m is odd), then an irreducible, unitary representation of G, $\pi = \theta(\sigma, \psi)$, can be constructed explicitly, and π corresponds to σ by Howe duality (i.e., $\mathrm{Hom}_{\overline{\mathrm{SL}}(2,F)\times G}(\omega_\psi \otimes \sigma, \pi) \neq 0$). We explain the construction of $\theta(\sigma, \psi)$. Write

$$L = Fe_2 + E + Fe_{-2}$$

where $e_{\pm 2}$ are isotropic, $(e_2, e_{-2}) = 1$ and E is the orthogonal complement in L of $Fe_2 + Fe_{-2}$. Put $X^{\pm} = Fe_{\pm 1} + Fe_{\pm 2}$, then $X^{\pm}$ are two dimensional isotropic subspaces, which are in duality under $(\ ,\ )$, and

$$X = X^+ + E + X^-. \tag{3.1}$$

(Note that our assumption $m \geqq 5$ implies $E \neq 0$.) We identify $X^{\pm} = F^2$ using the bases $\{e_1, e_2\}$, $\{e_{-2}, e_{-1}\}$, and we write the elements of G according to the decomposition (3.1). Let Q' be the parabolic subgroup of G, which preserves X^+, then Q' has Levi decomposition

$$Q' = (\mathrm{GL}(2, F)\cdot \mathrm{SO}(E))\cdot \mathscr{H}.$$

The Levi part is $\mathrm{GL}(2, F)\cdot \mathrm{SO}(E)$. We embed $\mathrm{GL}(2, F)$ in Q' by

$$r \mapsto \hat{r} = \begin{pmatrix} {}^t r^{-1} & & \\ & I_E & \\ & & wrw \end{pmatrix}, \qquad r \in \mathrm{GL}(2, F), \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\mathrm{SO}(E)$ is embedded by

$$h \mapsto \begin{pmatrix} I_2 & & \\ & h & \\ & & I_2 \end{pmatrix}, \qquad h \in \mathrm{SO}(E).$$

The unipotent radical $\mathscr{H}$ is isomorphic to the Heisenberg group corresponding to the symplectic space $Y \otimes E \cong E^2$. The elements of $\mathscr{H}$ may be realized using the above identifications as

$$\eta(u; z) = \begin{pmatrix} I_2 & u^* & z \\ & I_E & u \\ & & I_2 \end{pmatrix}$$

where

$$u = (u_2, u_1) \in E^2, \qquad u^* = \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix};$$

$u_i^* \in E^*$ sends e in E to $-(e, u_i)$. $z \in M(2, F)$ satisfies

$$w^t z + zw + w((u_i, u_j))_{i,j \leq 2} w = 0.$$

Consider E^2 as a symplectic space, with the symplectic form

$$\langle (u_2, u_1), (u_2', u_1') \rangle = (u_1, u_2') - (u_2, u_1').$$

The Heisenberg group of the symplectic space E^2 is $E^2 \oplus F$ (as a set) with multiplication law

$$(u, t) \cdot (u', t') = (u + u', t + t' + \tfrac{1}{2}\langle u, u' \rangle).$$

To see that $\mathscr{H}$ is isomorphic to the Heisenberg group of E^2, consider the map $B : \mathscr{H} \to E^2 \oplus F$ defined by

$$B(\eta(u; z)) = \left(uw, \tfrac{1}{2}\mathrm{tr} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (zw + \tfrac{1}{2}((u_i, u_j))_{i,j \leq 2}) \right).$$

This is an isomorphism, and it commutes the action of $\mathrm{SL}(2, F) \cdot \mathrm{SO}(E)$ on $\mathscr{H}$ by conjugation with the natural right action of $\mathrm{SL}(2, F) \cdot \mathrm{SO}(E)$ (as a subgroup of $\mathrm{Sp}(E^2)$ on the Heisenberg group of E^2). Put

$$Q = \mathrm{GL}(2, F) \cdot \mathscr{H}, \qquad D = \mathrm{SL}(2, F) \cdot H.$$

Let ρ_ψ be the unique, irreducible representation of $\mathscr{H}$, which has central character ψ. (The center of $\mathscr{H}$ consists of the elements $\eta(0, z)$.) ρ_ψ can be extended to a representation of $\overline{\mathrm{SL}}(2, F) \cdot \mathscr{H}$ by $\omega_\psi' \rho_\psi$, where ω_ψ' is the (restriction to $\overline{\mathrm{SL}}(2, F)$ of the) Weil representation corresponding to the dual pair $(\mathrm{SL}(2, F), \mathrm{SO}(E))$. The representation $\theta(\sigma, \psi)$ constructed in [S] satisfies

(3.2) $$\mathrm{Res}_Q\, \theta(\sigma, \psi) = \mathrm{Ind}_D^Q\, \omega_\psi' \rho_\psi \otimes \sigma.$$

By Mackey theory, this representation is irreducible, and uniquely determined by σ. To obtain a realization of (3.2), we realize ω_ψ (the Weil representation corresponding to the pair $(\mathrm{SL}(2,F), G)$), using a different polarization of $Z = Y \otimes X$. This time we polarize

$$Z = Z^+ + Z^-$$

where

$$Z^\pm = Y \otimes X^\pm + Y^\pm \otimes E$$

and realize ω_ψ in $L^2(Z^-)$, the subspace of smooth vectors being $S(Z^-)$. We identify $Z^- = M(2,F) \otimes E$. Let $f \in S(Z^-)$ and $\xi \in H_\sigma$ (the Hilbert space of σ). Define for $b \in Q$, $e \in E$,

$$I_{f,\xi}(b)(e) = \int_{\mathrm{SL}(2,F)} \omega_\psi(g,b) f(w;e)\sigma(g)\xi \, dg. \tag{3.3}$$

Then the integral (3.3) converges absolutely. For $b \in Q$, $I_{f,\xi}(b)$ is an L^2-function on E, with values in H_σ. For $\eta \in \mathscr{H}$ and $r \in \mathrm{SL}(2,F)$, we have

$$I_{f,\xi}(\eta \hat{r} b) = (\rho_\psi(\eta)\omega'_\psi(r,1) \otimes \sigma(r)) I_{f,\xi}(b), \qquad b \in Q.$$

$I_{f,\xi}$ is an L^2-function. More precisely, the function (on $\mathrm{GL}(2,F)$)

$$r \mapsto \int_E \| I_{f,\xi}(\hat{r})(e) \|^2_{H_\sigma} de$$

is $\mathrm{SL}(2,F)$-left invariant, and (the L^2-norm of $I_{f,\xi}$ which equals)

$$\begin{aligned} &\int_{\mathrm{SL}(2,F)\backslash \mathrm{GL}(2,F)} \int_E \| I_{f,\xi}(\hat{r})(e) \|^2_{H_\sigma} de\, dr \\ &= \int_{\mathrm{SL}(2,F)} (\sigma(g)\xi, \xi)(\omega_\psi(g,1) f, f) dg. \end{aligned} \tag{3.4}$$

All last three integrals converge absolutely. Now, we may extend the action of Q on $V'(\sigma,\psi) = \mathrm{Span}\{I_{f,\xi} \mid f \in S(Z^-),\ \xi \in H_\sigma\}$ to an action of G on $V'(\sigma,\psi)$ by

$$h \cdot I_{r,\xi} = I_{\omega_\psi(1,h) f, \xi}, \qquad h \in G.$$

By (3.4), we see that this action is well defined and unitary, and hence extends to the full space $\overline{V'(\sigma,\psi)}$. This realizes $\theta(\sigma,\psi)$. Note that $T(f,\xi) = I_{f,\xi}$ defines a nonzero element of $\mathrm{Hom}_{\overline{\mathrm{SL}(2,F)} \times G}(\omega_\psi^\infty \otimes \sigma^\infty, \theta(\sigma,\psi))$.

Let $V(\sigma,\psi) = \mathrm{Span}\{I_{f,\xi} \mid f \in S(Z^-), \xi \in H_\sigma^\infty\}$. This is an algebraically irreducible subspace of $\theta(\sigma,\psi)$, and on it we can define a nonzero linear functional γ, satisfying (2.1) (for suitable l_0). Assume that σ has a Whittaker model with respect to U and the character $\psi^{\lambda/2}$, where $\lambda \in F^*$. Let α be a corresponding Whittaker functional on H_σ^∞. Define on $V(\sigma, \psi)$ a linear functional γ_λ by

$$\gamma_\lambda(I_{f,\xi}) = \alpha(I_{f,\xi}(\hat{l}_\lambda)(0)) = \int_{SL(2,F)} \omega_\psi(g, 1) f(w; 0)\alpha(\sigma(g)\xi)dg.$$

Let $l_\lambda = \frac{1}{2}\lambda e_2 + e_{-2}$, then $(l_\lambda, l_\lambda) = \lambda$, and it is easy to check that γ_λ (is nonzero and) satisfies (2.1) with $l_0 = l_\lambda$. By Proposition 2.1 and Theorem 2.2 γ_λ is the unique, up to scalar multiples, linear functional on (the smooth vectors of) $\theta(\sigma, \psi)$, satisfying (2.1), and it automatically satisfies

$$\gamma_\lambda(\theta(\sigma, \psi)(\delta)\varphi) = \gamma_\lambda(\varphi), \qquad \delta \in O^c_{l_\lambda}.$$

REMARKS. (1) Recently Waldspurger has given a proof of the Howe duality conjecture [W], and hence $\theta(\sigma, \psi)$ described above is the theta-lift of σ to G. See also [M.V.W] where a proof is given for unramified dual pairs.

(2) An explicit description of the Howe lift in case of a general dual pair in the stable range has been obtained recently by Li [L]. It is similar to the description above.

§4. Nondegenerate representations

We keep the notation of Section 3. In this section we introduce nondegenerate representations of G and describe some elementary properties of these representations.

Let (π, H_π) be an irreducible, unitary representation of G. H_π denotes a Hilbert space realization of the action of π.

DEFINITION 4.1. π is called degenerate if the spectral measure of $\mathrm{Res}_N \pi$ is concentrated on characters of N of the form $\psi_l(\eta(v)) = \psi((v, l))$, where $l \in L$ is isotropic. We call such characters degenerate. (Otherwise, we call π nondegenerate.)

LEMMA 4.2. *Assume that there is a nonzero linear functional γ on H_π^∞ satisfying*

$$\gamma(\pi(\eta(v))\zeta) = \psi((v, l_0))\gamma(\zeta), \qquad \zeta \in H_\pi^\infty$$

where $(l_0, l_0) \neq 0$, then π is nondegenerate.

PROOF. Assume that π is degenerate. Let $\zeta \in H_\pi^\infty$, and let f_ζ be a representing function on L corresponding to the decomposition

$$H_\pi = \int_L H_{\pi,l} d\mu(l)$$

(N acts on $H_{\pi,l}$ according to ψ_l). Let ϕ be in $C_c^\infty(L)$, then

$$\int_L \phi(v)\pi(\eta(v)) f_\zeta(l) dv = \hat{\phi}(l) f_\zeta(l).$$

Hence if $\hat{\phi}$ is supported away from the isotropic vectors of L, then

$$\hat{\phi}(l) f_\zeta(l) = 0$$

for almost all l. Thus, for such ϕ and $\zeta \in H_\pi^\infty$,

$$\int_L \phi(v)\pi(\eta(v))\zeta \, dv = 0. \tag{4.1}$$

Applying γ to (4.1), we get

$$\hat{\phi}(l_0)\gamma(\zeta) = 0$$

for all ϕ as above. This implies that $\gamma(\zeta) = 0$ for $\zeta \in H_\pi^\infty$, a contradiction. □

THEOREM 4.3. *Let π be an infinite dimensional, irreducible, unitary representation of G. Assume that π is nondegenerate, then in the spectral measure of* $\mathrm{Res}_N \pi$, *the measure of the set of degenerate characters of N is zero.*

PROOF. The proof is essentially a repetition of the proof in the appendix in [PS]. We use the notation of Section 3.

Since π is infinite dimensional, the Howe–Moore theorem [HM] implies that π has no $Z(\mathscr{H})$ fixed vectors where $Z(\mathscr{H})$ is the center of $\mathscr{H}$, which is isomorphic to the Heisenberg group of E^2. By Mackey theory, there is a unitary representation σ of $\overline{\mathrm{SL}}(2, F)$, such that

$$\mathrm{Res}_Q \pi = \mathrm{Ind}_D^Q \omega'_\psi \rho_\psi \otimes \sigma$$

and $\omega'_\psi \otimes \sigma$ factors to $\mathrm{SL}(2, F)$. Since D is normal in Q,

$$\begin{aligned}\mathrm{Res}_N \pi &= \mathrm{Res}_N \mathrm{Ind}_D^Q \omega'_\psi \rho_\psi \otimes \sigma \\ &= \int_{F^*} \mathrm{Res}_N(\omega'_\psi \rho_\psi \otimes \sigma)^t d^*t \\ &= \int_{F^*} \mathrm{Res}_N(\omega'_{\psi^t} \rho_{\psi^t} \otimes \sigma^t) d^*t.\end{aligned}$$

Here $(\quad)^t$ denotes conjugation by

$$\begin{pmatrix} t & & \\ & I_L & \\ & & t^{-1} \end{pmatrix};$$

σ^t is σ conjugated by $\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right)$ and $\psi^t(x) = \psi(tx)$. Thus we need to examine the spectral decomposition of representations of D which have a nontrivial central character, i.e., of the form $\omega'_{\psi_1}\rho_{\psi_1} \otimes \delta$, where δ is as σ above and ψ_1 is a nontrivial character of F. Assume that $\overline{\mathrm{SL}}(2, F) = \widetilde{\mathrm{SL}}(2, F)$, the metaplectic group, then δ is a genuine representation of $\widetilde{\mathrm{SL}}(2, F)$. Since $\mathrm{Ind}_D^Q\, \omega'_\psi \rho_\psi \otimes -$ preserves direct integrals, then (decomposing σ into a direct integral of irreducible (genuine) unitary representations), we may assume that δ is irreducible. Since δ is genuine, it is infinite dimensional. We have the following lemma:

LEMMA. $\mathrm{Res}_N(\omega'_{\psi_1}\rho_{\psi_1} \otimes \delta)$ *has a spectral measure concentrated on nondegenerate characters of* N.

Given this lemma, the theorem is proved in case $\overline{\mathrm{SL}}(2, F) = \widetilde{\mathrm{SL}}(2, F)$. Now assume that $\overline{\mathrm{SL}}(2, F) = \mathrm{SL}(2, F)$. Let us decompose $\sigma = \sigma_1 \oplus \sigma_2$ where σ_1 is a multiple of the trivial representation of SL(2, F) and σ_2 has no invariant vector, then

$$\mathrm{Res}_Q\, \pi = \mathrm{Ind}_D^Q\, \omega'_\psi \rho_\psi \otimes \sigma_1 \oplus \mathrm{Ind}_D^Q\, \omega'_\psi \rho_\psi \otimes \sigma_2. \tag{4.2}$$

Decomposing σ_2 into a direct integral of (infinite dimensional) irreducible, unitary representations of $\mathrm{SL}(2, F)$, it will be clear by the proof of the lemma that $\mathrm{Res}_N\, \mathrm{Ind}_D^Q\, \omega'_\psi \rho_\psi \otimes \sigma_2$ has spectral measure concentrated on nondegenerate characters of N. Thus, if $\sigma_1 = 0$, we are done. Assume that $\sigma_1 \neq 0$, then it will be clear by the proof of the lemma that the spectral measure of $\mathrm{Res}_N\, \mathrm{Ind}_D^Q\, \omega'_\psi \rho_\psi \otimes \sigma_1$ is concentrated on the set C of degenerate characters of N. Thus if P_π denotes the projection valued measure on $\hat{N}$ associated with π, then $P_\pi(C)$ is the projection on the first summand in (4.2), and hence $P_\pi(C)$ commutes with $\pi(Q)$. The Levi subgroup M of P acts naturally on $\hat{N}$, and for $m \in M$ and a Borel subset $W \subset \hat{N}$, one has

$$P_\pi(m \cdot W) = \pi(m) P_\pi(W) \pi(m)^{-1}$$

($m\,\cdot$ –denotes the action of M on $\hat{N}$). Since C is M-invariant we find that $P_\pi(C)$

commutes with $\pi(M)$. Since G is generated by M and Q, $P_\pi(C)$ commutes with $\pi(G)$, and hence $H_\pi = P_\pi(C)$. This implies that π is degenerate, contrary to our assumption. Thus $\sigma_1 = 0$, and the theorem is proved. □

Proof of the Lemma. We use the notation of Section 3. An element of N can be written as

$$\eta(v) = \eta(v')\widehat{u(t)}\eta\left(0; x\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

for $v = xe_2 + v' + te_{-2}$, where $v' \in E$, $t, x \in F$ (recall that $u(t) = \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$). ρ_{ψ_1} is realized in $L^2(E)$, so that for $\phi \in S(E)$,

$$\begin{aligned} \omega'_{\psi_1}\rho_{\psi_1}(\eta(v))\phi(e) &= \psi_1^{-1}((v', e))\psi_1(x)\omega'_{\psi_1}(u(t), 1)\phi(e) \\ &= \psi_1(x - (v', e) - \tfrac{1}{2}t(e, e))\phi(e) \\ &= \psi_1((-\tfrac{1}{2}(e, e)e_2 - e + e_{-2}, v))\phi(e). \end{aligned}$$

Thus the spectral measure of $\omega'_{\psi_1}\rho_{\psi_1}$ is concentrated on $\psi_1((\cdot, l_e))$ where $l_e = -\frac{1}{2}(e, e)e_2 - e + e_{-2}$, $e \in E$. We may take the spectral measure to be the Haar measure de. (Note that $(l_e, l_e) = 0$.) Since δ is infinite dimensional, the spectral measure of δ as a representation of $\overline{\mathrm{SL}}(2, F)\cdot\mathscr{H}$ is concentrated on characters of the form $\psi_1(at) = \psi_1((v, ae_2))$, where a lies in a union of cosets of F modulo $(F^*)^2$. The measure may be taken to be the Haar measure d^*a. Thus the spectral measure of $\mathrm{Res}_N(\omega'_{\psi_1}\rho_{\psi_1} \otimes \delta)$ is concentrated on characters of the form $\psi_1((\cdot, l_{e,a}))$, where $l_{e,a} = (a - \frac{1}{2}(e, e))e_2 - e + e_{-2}$, with measure ded^*a. Since $(l_{e,a}, l_{e,a}) = 2a$, then $l_{e,a}$ is isotropic if and only if $a = 0$. The set of vectors $l_{e,0}$ has measure zero. (Note that if δ is trivial then the spectral measure is concentrated on the characters $\psi_1((\cdot, l_e))$, which are all degenerate.) The lemma is proved. □

Let us identify $\hat{N}$ with L by $\psi_l \leftrightarrow l$. Representatives of the orbits of M in $\hat{N}$ can be chosen as follows: $I = \eta(0), \eta(l_0), \eta(l_1), \ldots, \eta(l_r)$, where l_0 is a nonzero isotropic vector and $(l_i, l_i) = \lambda_i$, $i = 1, \ldots, r$ such that $\{\lambda_1, \ldots, \lambda_r\}$ is a set of representatives of $(F^*)^2 \backslash F^*$. We choose

$$l_0 = e_{-2},$$

$$l_i = e_2 + \frac{\lambda_i}{2}e_{-2}, \qquad i = 1, \ldots, r.$$

Now we can formulate

COROLLARY 4.4. *Let π be as in Theorem 4.3, then there is a subset $T(\pi)$ of $\{1, \dots, r\}$, and for each i in $T(\pi)$, there is a unitary representation τ_i of O_{l_i} such that*

$$\operatorname{Res}_P \pi = \bigoplus_{i \in T(\pi)} \operatorname{Ind}^P_{O_{l_i} N} \tau_i \otimes \psi_{l_i}. \tag{4.3}$$

PROOF. By Theorem 4.3, there is a subset $T(\pi)$ of $\{1, \dots, r\}$ such that

$$H_\pi = \bigoplus_{i \in T(\pi)} P_\pi(M \cdot l_i). \tag{4.4}$$

$M \cdot l_i$ denotes the orbit of $\eta(l_i)$ under M. Put $H_{\pi,i} = P_\pi(M \cdot l_i)$ then $H_{\pi,i}$ is a P-invariant subspace of H_π, and the representation of P in $H_{\pi,i}$ is supported on one orbit of characters of N. By Mackey theory, the representation of P in $H_{\pi,i}$ is of the form $\operatorname{Ind}^P_{O_{l_i} N} \tau_i \otimes \psi_{l_i}$, where τ_i is a unitary representation of O_{l_i}. □

Before we continue our study of nondegenerate representations we need the following useful lemmas.

Put

$$B_1 = \left\{ \left[\begin{array}{c|c|c|c|c} x & & & & \\ \hline & y & * \cdots * & * & \\ \hline & & & * & \\ & & h & \vdots & \\ & & & * & \\ \hline & & & y^{-1} & \\ \hline & & & & x^{-1} \end{array} \right] \in G \right\},$$

$$B_0 = \left\{ \left[\begin{array}{c|c|c|c|c} x & & & & \\ \hline & y & * \cdots * & * & \\ \hline & & & * & \\ & & I_E & \vdots & \\ & & & * & \\ \hline & & & y^{-1} & \\ \hline & & & & x^{-1} \end{array} \right] \in G \right\}$$

(zeros elsewhere).

Lemma 4.5. *$M = \overline{B_1 O^c_{l_i}}$, $i = 0, \ldots, r$. ($O^c_{l_0}$ denotes the stabilizer in N of $\eta(l_0) = \eta(e_{-2})$.)*

Proof. $B_1 \backslash M$ is isomorphic to the variety L_0 of isotropic lines in L by the map $B_1 m \mapsto F m^{-1} e_2$. $O^c_{l_i}$ acts on L_0 from the right by $(Fe) \cdot h = F(h^{-1}e)$. The orbits of $O^c_{l_i}$ in L_0 correspond to the double cosets of $B_1 \times O^c_{l_i}$ in M. Consider the open dense subset

$$\{Fl \in L_0 \mid (l, l_i) \neq 0\}.$$

It is easy to see that this set forms one orbit in L_0 under $O^c_{l_i}$. Fe_2 is a representative for this orbit and it corresponds to the double coset $B_1 O^c_{l_i}$. □

Corollary 4.6. *$M = \overline{B_0 O^c_{l_i}}$, $i = 0, \ldots, r$.*

Proof. Since $B_1 = B_0 \cdot \mathrm{SO}(E)$

$$\left(\mathrm{SO}(E) = \left\{ \begin{pmatrix} I_2 & & \\ & h & \\ & & I_2 \end{pmatrix} \in G \right\} \right),$$

and $\mathrm{SO}(E) l_i = l_i$, then $M = \overline{B_1 O^c_{l_i}} = \overline{B_0 \,\mathrm{SO}(E) O^c_{l_i}} = \overline{B_0 O^c_{l_i}}$. □

Corollary 4.7. *$M = \overline{B_0 O^c_l}$ for all $l \in L$ outside a subset of codimension one in L. (For l isotropic, O^c_l denotes the stabilizer in M of $\eta(l)$.)*

Proof. Let $l \in L$ be nonzero. If there is b in B_0 such that for some $0 \leqq i \leqq r$, $b\eta(l_i)b^{-1} = \eta(l)$, then this claim follows from Corollary 4.6. Such b exists iff $(l, e_2) \neq 0$. Indeed, we may assume that $(l, l) = (l_i, l_i)$ for some $0 \leqq i \leqq r$, and let us look for b as above of the form

$$\begin{pmatrix} 1 & & \\ & b' & \\ & & 1 \end{pmatrix},$$

and so we must have

$$b' \cdot l_i = l,$$

$$b' \cdot e_2 = y e_2 \qquad \text{for some } y \in F^*,$$

$$b' \cdot z \in z + F e_2 \qquad \text{for all } z \in E.$$

This happens iff

$$(l_i, e_2) = y(l, e_2)$$

and

$$(l, z) = \alpha_z(l, e_2) \qquad \text{for some } \alpha_z \in F.$$

Since $(l_i, e_2) \neq 0$ for $0 \leqq i \leqq r$, we must have $(l, e_2) \neq 0$. This defines y and α_z. Now it is clear that $M = \overline{B_0 O_l^c}$ iff l is in the set $\{l \in L(l, e_2) \neq 0\}$. □

LEMMA 4.8. *Let $\{m_\nu\}$ be a sequence in M such that*

$$m_\nu \eta(l_i) m_\nu^{-1} \xrightarrow[\nu \to \infty]{} \eta(l_j), \qquad i, j = 1, \ldots, r$$

then $i = j$ and $m_\nu \xrightarrow[\nu \to \infty]{} I \pmod{O_{l_i}}$.

PROOF. Write

$$m_\nu = \begin{pmatrix} x_\nu & & \\ & h_\nu & \\ & & x_\nu^{-1} \end{pmatrix},$$

then $x_\nu h_\nu l_i \xrightarrow[\nu \to \infty]{} l_j$ and hence $x_\nu^2 (l_i, l_i) \xrightarrow[\nu \to \infty]{} (l_j, l_j)$, that is $x_\nu^2 \lambda_i \xrightarrow[\nu \to \infty]{} \lambda_j$. This implies that $i = j$ and $x_\nu^2 \xrightarrow[\nu \to \infty]{} 1$. Indeed, let K' be large (such that $\lambda_j^{-1} \mathscr{P}^{K'} = \mathscr{P}^K$, $K \geqq 1$), and

$$x_\nu^2 \lambda_i \in \lambda_j + \mathscr{P}^{K'} = \lambda_j (1 + \mathscr{P}^K).$$

Since $1 + \mathscr{P}^K \subset (F^*)^2$, we find that $i = j$, and $x_\nu^2 \in 1 + \mathscr{P}^K$. By Corollary 4.6 we may assume that $m_\nu = b_\nu \in B_0$. Write

$$b_\nu = \begin{bmatrix} x_\nu & & & & \\ & y_\nu & & & \\ & & I_E & & \\ & & & y_\nu^{-1} & \\ & & & & x_\nu^{-1} \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & 1 & e_\nu^* & -\frac{1}{2}(e_\nu, e_\nu) & \\ & & I_E & e_\nu & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

then

$$b_\nu \eta(l_i) b_\nu^{-1} \eta(-l_i)$$

$$= \eta\left(\left(x_\nu y_\nu \left(1 - \frac{\lambda_i}{4}(e_\nu, e_\nu)\right) - 1\right) e_2 + \frac{x_\nu \lambda_i}{2} e_\nu + \frac{\lambda_i}{2}(x_\nu y_\nu^{-1} - 1) e_{-2}\right).$$

Thus, we conclude that

$$e_\nu \xrightarrow[\nu\to\infty]{} 0, \quad x_\nu y_\nu^{-1} \xrightarrow[\nu\to\infty]{} 1 \quad \text{and} \quad x_\nu y_\nu \xrightarrow[\nu\to\infty]{} 1.$$

Assume for simplicity that F is of odd residual characteristic. Let K be large and ν such that $x_\nu^2 \in 1 + \mathscr{P}^K$, then either $x_\nu \in 1 + \mathscr{P}^K$ or $x_\nu \in -1 + \mathscr{P}^K$. This implies that respectively $y_\nu \in 1 + \mathscr{P}^K$ or $y_\nu \in -1 + \mathscr{P}^K$. Thus, modulo O_{l_i}, b_ν is close to I. A similar argument holds if F of even residual characteristic. □

We now get back to nondegenerate representations. Let π be as in Theorem 4.3, and recall the decompositions (4.3), (4.4).

Proposition 4.9. *Let V be an N-invariant subspace of H_π^∞, then*

$$\bar V = \bigoplus_{i\in T(\pi)} \bar V \cap H_{\pi,i}.$$

Also

$$\bar V \cap H_{\pi,i} = \overline{V \cap H_{\pi,i}}. \tag{4.5}$$

Proof. Let $l \in L$ be nonisotropic and L_1 a large compact open subgroup of L. Define for $f \in V$

$$f_{l,L_1} = \frac{1}{\mu(L_1)} \int_{L_1} \psi^{-1}((v, l))\pi(\eta(v)) f\, dv.$$

Since f is smooth, the integral reduces to a finite sum and $f_{l,L_1} \in V$. ($\mu(L_1)$ is the measure of L_1.) Write $f = \Sigma_{i\in T(\pi)} f^{(i)}$, with $f^{(i)} \in H_{\pi,i}$. Note that $f^{(i)}$ is a P smooth vector, i.e., a smooth function on P (see (4.3)). For

$$m = \begin{pmatrix} x & & \\ & h & \\ & & x^{-1} \end{pmatrix},$$

put

$$\tilde m = \begin{pmatrix} x^{-1} & & \\ & h & \\ & & x \end{pmatrix}.$$

We have

$$f_{l,L_1}^{(i)}(m)=\frac{1}{\mu(L_1)}\int_{L_1}\psi_l^{-1}(\eta(v))f^{(i)}(m\eta(v)m^{-1}m)dv$$
$$=\frac{1}{\mu(L_1)}\int_{L_1}\psi_l^{-1}(\eta(v))\psi_{l_i}(m\eta(v)m^{-1})dv\, f^{(i)}(m)$$
$$=\frac{1}{\mu(L_1)}\int_{L_1}\psi^{-1}((v,l-\tilde{m}^{-1}\cdot l_i))dv\cdot f^{(i)}(m).$$

Here $\tilde{m}^{-1}\cdot l_i=xh^{-1}l_i$. Assume that $l=m_l^{-1}\cdot l_{j_0}$, $j_0=1,\dots,r$, then taking L_1 large enough (depending on l), we see using Lemma 4.8 (and its proof) that in order that the last integral be nonzero, we must have $i=j_0$, and that in this case the integral defines the characteristic function of a small neighbourhood U_{l,L_1} of $\tilde{m}_l$ in $O_l\backslash M$. Since $f_{l,L_1}=\Sigma f_{l,L_1}^{(i)}$, we get that

$$f_{l,L_1}=\chi_{U_{l,L_1}}\cdot f^{(j_0)},\qquad \text{where } l\in M\cdot l_{j_0}. \tag{4.6}$$

This implies that $\chi_{U_{l,L_1}}\cdot f^{(j_0)}\in V$ for large compact open subgroups L_1. From this we conclude that $f^{(j_0)}\in\bar{V}\cap H_{\pi,j_0}$. This proves the first assertion. The second assertion is also an easy consequence of (4.6). □

Denote by $C_c^\infty(M,O_{l_i};H_{\tau_i})$ the subspace of smooth functions in $\mathrm{Ind}_{O_{l_i}N}^P\tau_i\otimes\psi_{l_i}$, which are compactly supported modulo $O_{l_i}N$. (We consider their restriction to M; H_{τ_i} is the Hilbert space of τ_i.) Let P_i denote the projection of H_π on $H_{\pi,i}$. The proof of Proposition 4.9 shows that for $i\in T(\pi)$

$$P_i(H_\pi^\infty)\cap C_c^\infty(M,O_{l_i};H_{\tau_i})\subset H_\pi^\infty\cap H_{\pi,i}. \tag{4.7}$$

Put

$$V'_{\tau_i}=\mathrm{Span}\{f^{(i)}(m)\mid f\in H_\pi^\infty,\ m\in M\}$$
$$=\mathrm{Span}\{f^{(i)}(I)\mid f\in H_\pi^\infty\},$$
$$H'_{\tau_i}=\bar{V}'_{\tau_i}.$$

This is an O_{l_i}-invariant subspace of H_{τ_i}. Since $H_\pi^\infty\subseteq\bigoplus_{i\in T(\pi)}\mathrm{Ind}_{O_{l_i}N}^P H'_{\tau_i}\otimes\psi_{l_i}$, and H_π^∞ is dense in H_π, we find that $H'_{\tau_i}=H_{\tau_i}$. Let us show that $P_i(H_\pi^\infty)\cap C_c^\infty(M,O_{l_i};H_{\tau_i})$ is quite large.

Lemma 4.10. $C_c^\infty(M,O_{l_i};V'_{\tau_i})\subset P_i(H_\pi^\infty)\cap C_c^\infty(M,O_{l_i};H_{\tau_i})$.

Proof. Let $\xi_0\in V'_{\tau_i}$, then there is $f\in H_\pi^\infty$ such that $f^{(i)}(I)=\xi_0$. Since $f^{(i)}$ is a smooth function, there is a small neighbourhood U of I in M, such that $f^{(i)}(hu)=\tau_i(h)\xi_0$ for all h in O_{l_i} and $u\in U$. As in (4.6), $\chi_{O_{l_i}U}f^{(i)}\in P_i(H_\pi^\infty)$. From this we conclude that given $\xi_0 \in V'_{\tau_i}$ and $m_0 \in M$, there is a small neighbourhood of m_0, U_0, such that for every neighbourhood U of m_0, contained in U_0, there is $\varphi \in P_i(H_\pi^\infty)$ such that φ is supported in $O_{l_i}U$ and $\varphi(hu) = \tau_i(h)\xi_0$ for $h \in O_{l_i}$ and $u \in U$. The lemma is now clear. □

§5. Representations with the U-property

We keep the notation of the previous section. In this section we characterize representations of G satisfying the U property as those which are θ-lifts from $\overline{\mathrm{SL}}(2, F)$.

Let π be an irreducible, unitary, infinite dimensional representation of G.

Lemma 5.1. *If π satisfies the U-property, then*

$$\mathrm{Res}_P\, \pi = \bigoplus_{i \in T(\pi)} \mathrm{Ind}_{O_{l_i}N}^P\, \tau_i \otimes \psi_{l_i}$$

and $\mathrm{Res}_{O_{l_i}^c}\, \tau_i$ *is a multiple of the trivial representation of* $O_{l_i}^c$.

Proof. Note first that, by Lemma 4.2, π is nondegenerate. (We assume of course that functionals satisfying (2.1) exist for π.) Thus we have the decomposition (4.3). Define for $f \in H_\pi^\infty$ and $i \in T(\pi)$

$$\varphi^i(f) = f^{(i)}(I)$$

($f^{(i)}$ is a smooth function on P). φ^i is a linear map from H_π^∞ to H_{τ_i} satisfying

$$\varphi^i(\pi(\eta(v))f) = \psi_{l_i}(\eta(v))\varphi^i(f).$$

Let γ'_i be a linear functional on V'_{τ_i}, then $\gamma^i = \gamma'_i \circ \varphi^i$ is a linear functional on H_π^∞ satisfying

$$\gamma^i(\pi(\eta(v))f) = \psi_{l_i}(\eta(v))\gamma^i(f).$$

Since π satisfies the U-property,

$$\gamma^i(\pi(\delta)f) = \gamma^i(f) \qquad \forall \delta \in O_{l_i}^c.$$

That is

$$\gamma'_i(\tau_i(\delta)(f^{(i)}(I))) = \gamma'_i(f^{(i)}(I))$$

for all $\delta \in O_{l_i}^c$ and all linear functionals γ'_i on V'_{τ_i}. This implies

$$\tau_i(\delta)(f^{(i)}(I)) = f^{(i)}(I) \qquad \forall \delta \in O_{l_i}^c.$$

Thus $\mathrm{Res}_{O_{l_i}^c}\, \tau_i$ is trivial on V'_{τ_i} and hence trivial on $\overline{V'_{\tau_i}} = H_{\tau_i}$. □

The next lemma will be needed in Section 6.

LEMMA 5.2. *Assume that π does not satisfy the U-property, then there is a nonzero $f_0 \in H_\pi^\infty$ such that*

$$\gamma(f_0) = 0,$$

for any linear functional γ on H_π^∞ satisfying

$$\gamma(\pi(\delta\eta(v))f) = \psi_{l_0}(\eta(v))\gamma(f), \qquad v \in L, \quad \delta \in O_{l_0}^c.$$

Here $l_0 \in L$ is nonisotropic.

PROOF. Let γ_0 be a linear functional on H_π^∞ violating the U-property. We may assume that there is j_0, $1 \leqq j_0 \leqq r$ such that

$$\gamma_0(\pi(\eta(v))f) = \psi_{l_{j_0}}(\eta(v))\gamma_0(f)$$

but there are $\delta' \in O_{l_{j_0}}^c$ and $f' \in H_\pi^\infty$ such that

$$\gamma_0(\pi(\delta')f') \neq \gamma_0(f').$$

(The existence of γ_0 implies by Lemma 4.2 that π is nondegenerate.) Define on $V'_{\tau_{j_0}}$

$$\tilde{\gamma}_0(f^{(j_0)}(I)) = \gamma_0(f), \qquad f \in H_\pi^\infty.$$

This is a good definition. If $f^{(j_0)}(I) = 0$, then $f^{(j_0)}(m) = 0$ for m in a small neighbourhood U_{j_0} of I in $O_{l_{j_0}} \backslash M$, and hence $\chi_{U_{j_0}} f^{(j_0)} = 0$. We may assume that $U_{j_0} = U_{l_{j_0}, L_1}$ for a large enough compact open subgroup $L_1 \subset L$. By (4.6), we have

$$f_{l_{j_0}, L_1} = \chi_{U_{l_{j_0}, L_1}} f^{(j_0)} = 0$$

and hence

$$\gamma_0(f) = \gamma_0\left(\frac{1}{\mu(L_1)} \int_{L_1} \psi_{l_{j_0}}^{-1}(\eta(v))\pi(\eta(v))f\,dv\right) = \gamma_0(f_{l_{j_0}, L_1}) = 0.$$

Thus $\tilde{\gamma}_0$ is a linear functional on $V'_{\tau_{j_0}}$ for which

$$\tilde{\gamma}_0(\tau_{j_0}(\delta)\xi - \xi) \not\equiv 0, \qquad \xi \in V'_{\tau_{j_0}}, \quad \delta \in O_{l_{j_0}}^c.$$

Hence the subspace $V''_{\tau_{j_0}} = \mathrm{Span}\{\tau_{j_0}(\delta)\xi - \xi \mid \xi \in V'_{\tau_{j_0}}, \delta \in O_{l_{j_0}}^c\}$ is a nonzero ($O_{l_{j_0}}^c$-invariant) subspace of $V'_{\tau_{j_0}}$. Let f_0 be any element of $C_c^\infty(M, O_{l_{j_0}}; V''_{\tau_{j_0}})$. By Lemma 4.10 and (4.7) $f_0 \in H_\pi^\infty \cap H_{\pi, j_0}$. Let γ be a linear functional as in the statement of the lemma. Assume that $l_0 = m_0 \cdot l_{i_0}$, $(1 \leqq i_0 \leqq r)$. Define

$$\gamma_{m_0}(f) = \gamma(\pi(\tilde{m}_0)f)$$

where if

$$m_0 = \begin{pmatrix} x_0 & & \\ & h_0 & \\ & & x_0^{-1} \end{pmatrix},$$

then

$$\tilde{m}_0 = \begin{pmatrix} x_0^{-1} & & \\ & h_0 & \\ & & x_0 \end{pmatrix}.$$

We have

$$\gamma_{m_0}(\pi(\eta(v))f) = \psi(x_0^{-1}(h_0 v, l_0))\gamma_{m_0}(f) = \psi_{l_{i_0}}(\eta(v))\gamma_{m_0}(f).$$

As above, we may factor γ_{m_0} through $\varphi^{(i_0)}$,

$$\gamma_{m_0}(f) = \widetilde{\gamma_{m_0}}(f^{(i_0)}(I))$$

and so

$$\gamma(f) = \gamma_{m_0}(\pi(\tilde{m}_0^{-1})f) = \tilde{\gamma}_{m_0}(f^{(i_0)}(\tilde{m}_0^{-1})).$$

If $j_0 \neq i_0$, then since $f_0 \in H_\pi^\infty \cap H_{\pi,j_0}$, we get that $f_0^{(i_0)} = 0$ and hence $\gamma(f_0) = 0$. Assume that $j_0 = i_0$. Note that

$$\gamma_{m_0}(\pi(\delta)f) = \gamma_{m_0}(f), \qquad \delta \in O_{l_{i_0}}^c.$$

Since $f_0^{(i_0)}(\tilde{m}_0^{-1}) = f_0(\tilde{m}_0^{-1}) \in V''_{\tau_{i_0}}$, we get that $\gamma(f_0) = 0$. □

We are now ready to prove the main theorems of this section.

THEOREM 5.3. *Let (π, H_π) be an irreducible, unitary, infinite dimensional representation of G, with U-property. Then* $\mathrm{Res}_Q\, \pi$ *is irreducible.*

PROOF. Let H be a nonzero (closed) Q-invariant subspace of H_π. Denote by V the subspace of Q-smooth vectors of H. Let f be in V. Write $f = \Sigma_{i \in T(\pi)} f^{(i)}$ in the decomposition (4.4). By Corollary 4.6, $O_{l_i}^c B_0$ is an open dense subset of M. Thus (a representing function for) $f^{(i)}$ is determined by its restriction to $O_{l_i}^c B_0$ (its complement in M has measure zero). Since $B_0 \subset Q$, we find that $f^{(i)}$ is B_0-smooth, and by Lemma 5.1, $f^{(i)}$ is $O_{l_i}^c$-left invariant. Thus $f^{(i)}$ is a locally constant function in $H_{\pi,i}$. Thus $P_i(V)$ consists of locally constant functions. It is easy to see that Proposition 4.9 still holds for N-invariant, N-smooth subspaces V, such that $P_i(V)$ consists of locally constant functions, for $i \in T(\pi)$. Thus

$$\bar{V} = \bigoplus_{i \in T(\pi)} \bar{V} \cap H_{\pi,i} = \bigoplus_{i \in T(\pi)} \overline{V \cap H_{\pi,i}}.$$

Let i be such that $V \cap H_{\pi,i} \neq 0$, and define

$$W_i = \operatorname{Span}\{ f^{(i)}(m) \mid f^{(i)} \in V \cap H_{\pi,i}, m \in M \}.$$

This is an O_{l_i}-invariant subspace of H_{τ_i}, since $\tau_i(h)(f^{(i)}(m)) = f^{(i)}(hm)$. Denote the representation of O_{l_i} in $\bar{W}_i$ by σ_i, then clearly

$$\bar{V} \cap H_{\pi,i} \subseteq \operatorname{Ind}_{O_{l_i}N}^{P} \sigma_i \otimes \psi_{l_i}.$$

Let us show equality of the last two spaces. Let $\xi_0 \in W_i$ and $m_0 \in M$, such that $M = \overline{(m_0^{-1} O_{l_i}^c m_0) B_0}$. By Corollary 4.7, it is enough to consider such m_0. We claim that there is $f \in V \cap H_{\pi,i}$ such that $f(m_0) = \xi_0$. Indeed, there are $\varphi \in V \cap H_{\pi,i}$ and $m_1 \in M$, such that $\varphi(m_1) = \xi_0$. Let $\delta \in O_{l_i}^c$ and $b \in B_0$, be such that $m_0^{-1} m_1$ is close to $m_0^{-1} \delta m_0 b$, and

$$\varphi(m_0 m_0^{-1} m_1) = \varphi(m_0 m_0^{-1} \delta m_0 b).$$

Then

$$\begin{aligned} \xi_0 = \varphi(m_1) = \varphi(m_0 m_0^{-1} m_1) = \varphi(m_0 m_0^{-1} \delta m_0 b) = \varphi(\delta m_0 b) = (\text{Prop. 4.9}) \\ = \varphi(m_0 b) = (\pi(b)\varphi)(m_0). \end{aligned}$$

Since $B_0 \subset Q \cap P$, $\pi(b)\varphi \in V \cap H_{\pi,i}$. Take then $f = \pi(b)\varphi$. As in (4.7) and Lemma 4.10, we conclude that

$$C_c^\infty(M, O_{l_i}; W_i) \subset V \cap H_{\pi,i}$$

and hence

$$\operatorname{Ind}_{O_{l_i}N}^{P} \sigma_i \otimes \psi_{l_i} \subseteq \bar{V} \cap H_{\pi,i}.$$

Thus we proved that

$$\bar{V} = \bigoplus \bar{V} \cap H_{\pi,i} = \bigoplus \operatorname{Ind}_{O_{l_i}N}^{P} \sigma_i \otimes \psi_{l_i}.$$

In particular $\bar{V}$ is P-invariant. Since P and Q generate G, $\bar{V}$ is G-invariant, and hence $\bar{V} = H_\pi$. This proves the theorem. □

Theorem 5.4. *Let (π_1, H_{π_1}), (π_2, H_{π_2}) be two irreducible, unitary, infinite* *dimensional representations of G, with U-property. If* $\mathrm{Res}_Q\,\pi_1 \cong \mathrm{Res}_Q\,\pi_2$, *then* $\pi_1 \cong \pi_2$.

PROOF. Let $\alpha: H_{\pi_1} \to H_{\pi_2}$ be a Q-isomorphism. We claim that $T(\pi_1) = T(\pi_2)$ and that for $i \in T(\pi_1)$, $\alpha(H_{\pi_1,i}) = H_{\pi_2,i}$. Indeed, let $i \in T(\pi_1)$ and l in the orbit of l_i (under M). Let $f \in H_{\pi_1}^\infty$ and $L_1 \subset L$ a large compact open subgroup of L. Since $N \subset Q$, we have

$$\alpha(f_{l,L_1}) = \alpha(f)_{l,L_1}.$$

Now (4.6), (4.7) and Lemma 4.10 show that α sends $C_c^\infty(M, O_{l_i}, V'_{\tau_{1,i}})$ into $H_{\pi_2,i}$. ($H_{\pi_1,i} = \mathrm{Ind}_{O_{l_i}N}^P\,\tau_{1,i} \otimes \psi_{l_i}$.) Thus $\alpha(H_{\pi_1,i}) \subseteq H_{\pi_2,i}$. Similarly $\alpha^{-1}(H_{\pi_2,i}) \subseteq H_{\pi_1,i}$.

Let $f \in H_{\pi_1}^\infty$ and $i \in T(\pi_1) = T(\pi_2)$. As in the beginning of the proof of Theorem 5.3, we conclude that $\alpha(f^{(i)})$ is a locally constant function in $H_{\pi_2,i}$. Define then

$$T_i: H_{\pi_1}^\infty \to H_{\pi_2,i}$$

by

$$T_i(f) = \alpha(f^{(i)})(I).$$

We have

$$\begin{aligned} T_i(\pi_1(\eta(v))f) &= \alpha((\pi_1(\eta(v))f)^{(i)})(I) = \alpha(\pi_1(\eta(v))f^{(i)})(I) \\ &= \pi_2(\eta(v))\alpha(f^{(i)})(I) = \alpha(f^{(i)})(\eta(v)) = \psi_{l_i}(v)T_i(f). \end{aligned}$$

Composing T_i with any linear functional on $H_{\pi_2,i}$, and using the U-property of π_1, we find that

$$T_i(\pi_1(\delta)f) = T_i(f), \qquad \forall\,\delta \in O_{l_i}^c.$$

That is

$$\alpha(\pi_1(\delta)f^{(i)})(I) = \alpha(f^{(i)})(I) = \alpha(f^{(i)})(\delta), \qquad \forall\,\delta \in O_{l_i}^c.$$

Since α commutes with Q and $M = \overline{O_{l_i}^c B_0}$, we conclude that

$$\alpha(\pi_1(m)f^{(i)})(I) = \alpha(f^{(i)})(m) = \pi_2(m)\alpha(f^{(i)})(I), \qquad \forall\,m \in M.$$

This implies that $\mathrm{Res}_{H_{\pi_1,i}}\alpha$ commutes with M for $i \in T(\pi_1)$, and hence α commutes with M. Since Q and M generate G, α is a G-isomorphism. The theorem is proved. □

THEOREM 5.5. *Let (π, H_π) be an irreducible, unitary, infinite dimensional* *representation of G with U-property, then there is an irreducible, unitary representation σ of* $\overline{\mathrm{SL}}(2, F)$ *such that* $\pi = \theta(\sigma, \psi)$.

PROOF. As in the beginning of the proof of Theorem 4.3, there is a unitary representation σ of $\overline{\mathrm{SL}}(2, F)$ such that

$$\mathrm{Res}_Q \pi = \mathrm{Ind}_{\bar{B}}^{Q} \omega'_\psi \rho_\psi \otimes \sigma.$$

By Theorem 5.3, $\mathrm{Res}_Q \pi$ is irreducible, and so σ is irreducible. By (3.2) we have

$$\mathrm{Res}_Q \pi = \mathrm{Res}_Q \theta(\sigma, \psi).$$

By Theorem 2.2, $\theta(\sigma, \psi)$ satisfies the U-property, and hence by Theorem 5.4, $\pi \cong \theta(\sigma, \psi)$. □

COROLLARY 5.6. *Let* π *be as in Theorem* 5.5, *then the space of linear functionals on* H_π^∞ *satisfying* (2.1) *is at most one dimensional.*

PROOF. Follows from Theorem 5.5 and Proposition 2.1. □

§6. Special representations of rank one orthogonal groups

We use the notation of Section 1. We state our main theorem, which is the converse to Theorem 1.2.

THEOREM. *Special representations of* $G(\mathbb{A})$ *are obtained via the theta-correspondence from* $\overline{\mathrm{SL}}(2, \mathbb{A})$.

We first outline the proof and then give the details.

Note that since we assume that $\dim X \geqq 5$, then $G_v = G(k_v)$ is of rank at least two for almost all finite primes v (and if $\dim X \geqq 7$, then G_v is of rank two for all finite primes v). Hence the results of Sections 2–5 are valid for such G_v. Let π be an irreducible, cuspidal, automorphic representation of $G(\mathbb{A})$, and assume that π is special. We first note

LEMMA 6.1. π_v *satisfies the U-property for all finite primes* v, *with* $\mathrm{rank}(G_v) \geqq 2$.

Thus, by Theorem 5.5, for all such v, there is an irreducible, unitary representation σ_v of $\overline{\mathrm{SL}}(2, k_v)$, such that $\pi_v = \theta(\sigma_v, \psi_v)$ (ψ is a fixed nontrivial character of $k \backslash \mathbb{A}$). At this point we treat the case $m = 5$ separately.

PROOF OF THE THEOREM FOR THE CASE $m = 5$. Let $l \in L_k$ be nonzero and such that $\varphi_l \not\equiv 0$ for $\varphi \in \pi$, and hence $\varphi_l(\delta g) = \varphi_l(g)$ for $\delta \in O_l^c(\mathbb{A})$. This implies that

$$\int_{O_l^c(k)N(k)\backslash O_l^c(\mathbb{A})N(\mathbb{A})} \psi^{-1}((v,l))\varphi(\eta(v)\delta g)dv\, d\delta \not\equiv 0, \qquad \varphi \in \pi.$$

By Proposition 2.1 in [PS.S], it follows that the theta-lift of π (with respect to ψ) to $\widetilde{\mathrm{Sp}}(4,\mathbb{A})$ is nontrivial. If π does not lie in the image of theta-correspondence from $\widetilde{\mathrm{SL}}(2,\mathbb{A})$, then the theta-lift of π to $\widetilde{\mathrm{Sp}}(4,\mathbb{A})$ is cuspidal, and in this case, we may apply the main result of [PS.S] and obtain that there is an irreducible, automorphic, cuspidal representation π' of $\mathrm{SO}(3,2;\mathbb{A})$, which has a standard Whittaker model, and is locally equivalent to π at all places where G splits and π is unramified. For such a place v, we get that $\pi_v = \theta(\sigma_v, \psi_v)$ has a standard Whittaker model, which is impossible. Thus π is lifted (via theta) from $\widetilde{\mathrm{SL}}(2,\mathbb{A})$. □

From now on, we assume that $m \geqq 6$.

Next, we introduce certain Rankin–Selberg integrals for each nonzero $l \in L_k$. Let $l \in L_k$ be nonzero. Denote by H_l the stabilizer of l in G. If we denote by X_l the orthogonal complement of l in X, then $H_l = \mathrm{SO}(X_l)$. H_l is clearly a rank one orthogonal group defined over k. Let $P_l = M_l N_l$ be the parabolic subgroup of H_l, stabilizing the line through e_1.

$$M_l = \left\{ \begin{pmatrix} x & & \\ & h & \\ & & x^{-1} \end{pmatrix} \in G \;\middle|\; hl = l \right\} \cong \mathrm{GL}(1) \times O_l^c,$$

$$N_l = \{\eta(v) \mid (v,l) = 0\}.$$

In [R1], chapter 1, Rallis introduces a special Eisenstein series $E_l(f_s, h)$ on $H_l(\mathbb{A})$ corresponding to the representation $\mathrm{Ind}_{P_l(\mathbb{A})}^{H_l(\mathbb{A})} \mu_s$, where

$$\mu_s \begin{pmatrix} x & & \\ & h & \\ & & x^{-1} \end{pmatrix} = |x|^s \qquad \text{(unitary induction)},$$

where $hl = l$. Thus a function f_s in $\mathrm{Ind}\,\mu_s$ satisfies

$$f_s\left(\begin{pmatrix} x & * & * \\ & h & * \\ & & x^{-1} \end{pmatrix} g\right) = |x|^{s+(m-3)/2} f_s(g) \qquad (m-1 = \dim X_l).$$

The Eisenstein series

$$E_l(f_s, h) = \sum_{\gamma \in P_l(k)\backslash H_l(k)} f_s(\gamma h)$$

converges absolutely for $\mathrm{Re}(s) > (m-3)/2$, and continues to a meromorphic function on the complex plane. It has a simple pole at $s = (m-3)/2$, with constant residue. f_s is a certain special section which satisfies the following: assume that $f_s = \bigotimes f_{s,v}$, $f_{s,v}$ being unramified for almost all v (belonging to $\mathrm{Ind}_{P_l(k_v)}^{H_l(k_v)} \mu_{s,v}$), then for $f_{s,v}$ unramified, one has

$$f_{s,v}(I) = L_v\left(1, s + \frac{m-3}{2}\right) = (1 - q_v^{-s-(m-3)/2})^{-1}.$$

Also $L_v(1, s + (m-3)/2)^{-1} f_{s,v}$ is holomorphic for all v.

Consider the following Rankin–Selberg integral (for an irreducible, automorphic, cuspidal representation π' of $G(\mathbf{A})$,

$$I_l(\varphi, f_s) = \int_{H_l(k)\backslash H_l(\mathbf{A})} \varphi(h) E(f_s, h)\, dh, \qquad \varphi \in \pi'.$$

This is of course a meromorphic function, with poles contained in those of $E(f_s, h)$.

Lemma 6.2. *π' lifts via the theta-correspondence to $\overline{\mathrm{SL}}(2, \mathbf{A})$ iff there is $0 \neq l_0 \in L_k$ such that $I_{l_0}(\varphi, f_s)$ has a simple pole at $s = (m-3)/2$.*

Hence in order to achieve the proof of the theorem, we will show that for our special representation π there is a nonzero $l_0 \in L_k$ such that, on π, I_{l_0} has a simple pole at $s = (m-3)/2$. Choose any nonzero $l_0 \in L_k$ such that the Fourier coefficient $\varphi_{l_0}(g)$ is nontrivial on π.

Lemma 6.3. *Let φ be a cusp form in π, then (for* Re(*s*) *large)*

$$I_{l_0}(\varphi, f_s) = \int_{O_{l_0}^c(\mathbf{A}) N_{l_0}(\mathbf{A}) \backslash H_{l_0}(\mathbf{A})} \varphi_{l_0}(h) f_s(h)\, dh$$

(assuming that the measure of $O_{l_0}^c(k) \backslash O_{l_0}^c(\mathbf{A})$ is one).

Let Ω_0 be the set of archimedean primes, and those primes where G remains a rank one group, then for $v \notin \Omega_0$, $\pi_v = \theta(\sigma_v, \psi_v)$, and by Corollary 5.6, π_v^∞ has a unique (up to scalar multiples) linear functional γ_v, satisfying

$$\gamma_v(\pi_v(\eta(u))\zeta) = \psi_v((u, l_0))\gamma_v(\zeta), \qquad \zeta \in H_{\pi_v}^\infty.$$

Thus there is a unique right invariant subspace $W_{\psi_v}(\pi_v, l_0)$ of the space of smooth functions $w(g)$ on G_v, satisfying $w(\eta(u)g) = \psi_v((v, l_0))w(g)$, where π_v^∞ is realized by right translations. Assume that φ corresponds to a decomposable vector $\otimes\zeta_v$, and for $v \notin \Omega_0$, let w_{ζ_v} be the function in $W_{\psi_v}(\pi_v, l_0)$ corresponding to ζ_v, then clearly

$$\varphi_{l_0}(g) = \varphi_{l_0}(g_{\Omega_0}) \prod_{v \notin \Omega_0} w_{\zeta_v}(g_v)$$

and so, for $\mathrm{Re}(s)$ large (and $f_s = \otimes f_{s,v}$),

$$I_{l_0}(\varphi, f_s) = I_{\Omega_0, l_0}(\varphi, f_s) \prod_{v \notin \Omega_0} \int_{O_{l_0}^{\circ}(k_v)N_{l_0}(k_v)\backslash H_{l_0}(k_v)} w_{\zeta_v}(h) f_{s,v}(h)dh$$

where

$$I_{\Omega_0, l_0}(\varphi, f_s) = \int_{O_{l_0}^{\circ}(\mathbf{A}_{\Omega_0})N_{l_0}(\mathbf{A}_{\Omega_0})\backslash H_{l_0}(\mathbf{A}_{\Omega_0})} \varphi_{l_0}(g_{\Omega_0}) f_s(g_{\Omega_0})dg_{\Omega_0}.$$

Let Ω be a finite set of places containing Ω_0, outside of which, ψ_v, $\psi_{v_{l_0}}$, $f_{s,v}$ and ζ_v are unramified. For a finite place v_0 in Ω, we may choose f_{s,v_0} and ζ_{v_0} such that

$$I_{\Omega, l_0}(\varphi, f_s) = I_{\Omega - \{v_0\}, l_0}(\varphi, f_s).$$

(Indeed, fix ζ_v for $v \neq v_0$ and denote by $\varphi^{\zeta_{v_0}}$ the cusp form corresponding to $\otimes\zeta_v$. We may choose ζ_{v_0} such that

$$\varphi_{l_0}^{\zeta_{v_0}}\left(\begin{pmatrix} x_{v_0} & & \\ & I_L & \\ & & x_{v_0}^{-1} \end{pmatrix} g_{\Omega - \{v_0\}}\right)$$

vanishes outside a small neighbourhood of 1 in $k_{v_0}^*$ (independently of $g_{\Omega - \{v_0\}}$) and, in this neighbourhood, we have

$$\varphi_{l_0}^{\zeta_{v_0}}\left(\begin{pmatrix} x_{v_0} & & \\ & I_L & \\ & & x_{v_0}^{-1} \end{pmatrix} g_{\Omega - \{v_0\}}\right) = \varphi_{l_0}^{\zeta_{v_0}}(g_{\Omega - \{v_0\}}).$$

For this choose ζ'_{v_0} and take

$$\zeta_{v_0} = q_{v_0}^{-n \dim L} \int_{L(\mathscr{P}_{v_0}^{-n})} \psi_{v_0}^{-1}((u, l_0))\pi_{v_0}(\eta(u))\zeta'_{v_0} du$$

with n large enough (depending on ζ'_{v_0} only). Now choose f_{s,v_0} such that $f_{s,v_0}(I) = 1$ and so that its support lies in $P_{l_0}(k_{v_0})U_{v_0}$, where U_{v_0} is a small neighbourhood of I in $H_{l_0}(k_{v_0})$, which fixes ζ_{v_0}. With this choice, we can have the desired equality up to a constant.) For $v \notin \Omega$, let ζ_v^0 be the unramified vector with $w_{\zeta_v^0}(I) = 1$, and $f_{s,v}^0$ be unramified with

$$f_{s,v}^0(I) = L_v\left(1, s + \frac{m-3}{2}\right),$$

then we may choose data at Ω so that (for $\mathrm{Re}(s)$ large)

$$I_{l_0}(\varphi, f_s) = I_{\Omega_\infty, l_0}(\varphi, f_s) \prod_{v \notin \Omega} \int_{O_{l_0}^{\prime}(k_v)N_{l_0}(k_v)\backslash H_{l_0}(k_v)} w_{\zeta_v^0}(h) f_{s,v}^0(h) dh$$

where Ω_∞ denotes the set of archimedean primes of k. As in the proof of Theorem 2.1 in [PS.R], given $s_0 \in \mathbf{C}$, it is possible to choose data at Ω_∞ so that $I_{\Omega_\infty, l_0}(\varphi, f_s)$ is holomorphic and nonzero in a neighbourhood of s_0. (The criterion of [R.S] applies to show that $I_{\Omega_\infty, l_0}(\varphi, f_s)$ admits meromorphic continuation to $\mathbf{C}$.) We proved

Proposition 6.4. *Let Ω be a finite set of places containing Ω_0, outside of which ψ_v, $\psi_{v_{l_0}}$ and π_v are unramified. For $v \notin \Omega$, let ζ_v^0 be the unramified vector with $w_{\zeta_v^0}(I) = 1$, and $f_{s,v}^0$ the unramified vector with*

$$f_{s,v}^0(I) = L_v\left(1, s + \frac{m-3}{2}\right).$$

Then given $s_0 \in \mathbf{C}$, it is possible to choose the data $\zeta_v, f_{s,v}$ for $v \in \Omega$ so that for φ in π corresponding to $\bigotimes \zeta_v$ and $f_s = \bigotimes f_{s,v}$,

$$I_{l_0}(\varphi, f_s) = I_\infty(s) \prod_{v \notin \Omega} \int_{O_{l_0}^{\prime}(k_v)N_{l_0}(k_v)\backslash H_{l_0}(k_v)} w_{\zeta_v^0}(h) f_{s,v}^0(h) dh$$

for $\mathrm{Re}(s)$ *large and $I_\infty(s)$ is a meromorphic function which is holomorphic and nonzero in a neighbourhood of s_0.*

For this data, we will compute $I_{l_0}(\varphi, f_s)$ and show that it has a pole at $s = (m-3)/2$. This we will do by computing the local "unramified" integral

$$I_v^0(s)=\int_{O_{l_0}^c(k_v)N_{l_0}(k_v)\backslash H_{l_0}(k_v)} w_{\zeta_v^0}(h)f_{s,v}^0(h)dh$$

$$=L_v\left(1,s+\frac{m-3}{2}\right)\int_{k_v^*} w_{\zeta_v^0}\begin{pmatrix} x & & \\ & I & \\ & & x^{-1}\end{pmatrix}|x|^{s-(m-3)/2}d^*x$$

(by Iwasawa decomposition).

In the computation we will use the fact that $\pi_v=\theta(\sigma_v,\psi_v)$, and hence the space $W_{\psi_v}(\pi_v,l_0)$ is described by (2.5). The final computation will show

LEMMA 6.5. *With data as in Proposition* 6.4

$$I_{l_0}(\varphi,f_s)=I_\infty(s)L^\Omega\left(1,s+\frac{m+3}{2}\right)L^\Omega\left(1,s-\frac{m-5}{2}\right)F(\sigma,s)$$

where $F(\sigma,s)$ is a meromorphic function, which is nonzero at $s=(m-3)/2$.

This will finish the proof of the theorem. Now to the details.

PROOF OF LEMMA 6.1. Let v_0 be a finite prime, such that $\operatorname{rank}(G_{v_0})\geqq 2$ and π_{v_0} does not satisfy the U-property, then there is a vector $\zeta_0\in H^\infty_{\pi_{v_0}}$ as in Lemma 5.2. Choose nonzero $\zeta_v\in H^\infty_{\pi_v}$ for all v, such that $\zeta_{v_0}=\zeta_0$ and ζ_v is unramified for almost all v. Let φ be the cusp form in π corresponding to $\otimes\zeta_v$. Since π is special, we have for $0\neq l\in L_k$

$$\varphi_l(\delta g)=\varphi_l(g)\qquad\forall\delta\in O_l^c(\mathbf{A}).$$

The property of ζ_0 implies that

$$\varphi_l(p)=0\qquad\forall p\in P_{\mathbf{A}},\quad\forall l\in L_k.$$

Since $\varphi(g)=\Sigma_{l\in L_k}\varphi_l(g)$, we see that $\varphi\mid P_{\mathbf{A}}=0$. This implies that $\varphi=0$ since $P_k\backslash P_{\mathbf{A}}$ is dense in $G_k\backslash G_{\mathbf{A}}$, a contradiction. □

PROOF OF LEMMA 6.2. Taking the residue at $s=(m-3)/2$, we see that $I_{l_0}(\varphi,f_s)$ has a pole at $s=(m-3)/2$ if and only if

$$\int_{H_{l_0}(k)\backslash H_{l_0}(\mathbf{A})}\varphi(h)dh\not\equiv 0\qquad(\varphi\in\pi').$$

Assume that this condition holds for some nonzero $l_0\in L_k$. Let $\lambda=(l_0,l_0)$ then we claim that

$$\int_{k\backslash\mathbf{A}} \psi^{-1}\left(\frac{\lambda}{2}t\right) \int_{G_k\backslash G_{\mathbf{A}}} \theta_\psi^\phi\left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}, g\right) \varphi(g)dg\,dt \not\equiv 0$$

and hence π' lifts to $\overline{\mathrm{SL}}(2, \mathbf{A})$. Indeed, the last integral equals (using the standard polarization)

$$\int_{k\backslash\mathbf{A}} \psi^{-1}\left(\frac{\lambda}{2}t\right) \int_{G_k\backslash G_{\mathbf{A}}} \sum_{x\in X_k} \omega_\psi(1, g)\phi(x)\psi(\tfrac{1}{2}(x, x)t)\varphi(g)dg\,dt$$

$$= \int_{G_k\backslash G_{\mathbf{A}}} \sum_{\substack{x\in X_k\\ (x,x)=\lambda}} \omega_\psi(1, g)\phi(x)\varphi(g)dg$$

$$= \int_{G_k\backslash G_{\mathbf{A}}} \sum_{\gamma\in H_{l_0}(k)\backslash G_k} \omega_\psi(1, \gamma g)\phi(l_0)\varphi(g)dg$$

$$= \int_{H_{l_0}(k)\backslash G_{\mathbf{A}}} \omega_\psi(1, g)\phi(l_0)\varphi(g)dg$$

$$= \int_{H_{l_0}(\mathbf{A})\backslash G_{\mathbf{A}}} \omega_\psi(1, g)\phi(l_0)\left(\int_{H_{l_0}(k)\backslash H_{l_0}(\mathbf{A})} \varphi(hg)dh\right) dg.$$

It is easily seen that this integral is not identically zero. Conversely, assume that π' lifts to $\overline{\mathrm{SL}}(2, \mathbf{A})$, then there is a nonzero λ in k such that

$$\int_{k\backslash\mathbf{A}} \psi^{-1}\left(\frac{\lambda}{2}t\right) \int_{G_k\backslash G_{\mathbf{A}}} \theta_\psi^\phi\left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}, g\right) \varphi(g)dg\,dt \not\equiv 0.$$

The computation above shows that

$$\int_{H_{l_0}(k)\backslash H_{l_0}(\mathbf{A})} \varphi(h)dh \not\equiv 0 \qquad \text{for } l_0 \in L_k$$

such that $(l_0, l_0) = \lambda$. (Note that λ must be of this form.) □

PROOF OF LEMMA 6.3. (Expansion of $I_{l_0}(\varphi, f_s)$.) For Re(s) large enough, we have

$$I_{l_0}(\varphi, f_s) = \int_{P_{l_0}(k)\backslash H_{l_0}(\mathbf{A})} \varphi(h) f_s(h)dh = \int_{M_{l_0}(k)N_{l_0}(\mathbf{A})\backslash H_{l_0}(\mathbf{A})} \varphi^0(h) f_s(h)dh$$

where

$$\varphi^0(h) = \int_{N_{l_0}(k)\backslash N_{l_0}(\mathbf{A})} \varphi(nh)dn.$$

For $g \in G_{\mathbf{A}}$, the function $\varphi^0(\eta(v)g)$ is a function on $N_k N_{l_0}(\mathbf{A})\backslash N_{\mathbf{A}} \cong k\backslash\mathbf{A}$. We have the expansion

$$\varphi^0(g)=\sum_{t\in k^*}\varphi_{0,t}(g)$$

where

$$\varphi_{0,t}(g)=\int_{k\backslash\mathbf{A}}\psi^{-1}(tx)\varphi^0\left(\eta\left(\frac{x}{(l_0,l_0)}l_0\right)g\right)dx$$

$$=\int_{L_k\backslash L_{\mathbf{A}}}\psi^{-1}(t(v,l_0))\varphi(\eta(v)g)dv$$

$$=\varphi_{tl_0}(g)$$

($\varphi_{0,0}=0$ since φ is a cusp form). Thus

$$I_{l_0}(\varphi,f_s)=\int_{M_{l_0}(k)N_{l_0}(\mathbf{A})\backslash H_{l_0}(\mathbf{A})}\sum_{t\in k^*}\varphi_{tl_0}(h)f_s(h)dh$$

$$=\int_{O^c_{l_0}(k)N_{l_0}(\mathbf{A})\backslash H_{l_0}(\mathbf{A})}\varphi_{l_0}(h)f_s(h)dh$$

$$=\int_{O^c_{l_0}(\mathbf{A})N_{l_0}(\mathbf{A})\backslash H_{l_0}(\mathbf{A})}\varphi_{l_0}(h)f_s(h)dh.$$

We used the fact that π is special and hence $\varphi_{l_0}(\delta g)=\varphi_{l_0}(g)$ for $\delta\in O^c_{l_0}(\mathbf{A})$. Also we assume that the measure of $O^c_{l_0}(k)\backslash O^c_{l_0}(\mathbf{A})$ is one. □

Proof of Lemma 6.5. We have to compute $I^0_v(s)$. We know that $\pi_v=\theta(\sigma_v,\psi_v)$ for an irreducible unitary representation σ_v of $\overline{\mathrm{SL}}(2,F)$, which has a Whittaker model $\tilde{W}_{\psi_v}(\sigma_v,\frac{1}{2}(l_0,l_0))$ with respect to

$$U=\begin{pmatrix}1&0\\ *&1\end{pmatrix}$$

and the character $\psi^{(l_0,l_0)/2}$. Formula (2.5) describes $W_{\psi_v}(\pi_v,l_0)$ in terms of this Whittaker model of σ_v. In particular $w_{\zeta^0_v}$ is obtained as follows: let ϕ^0_v be the characteristic function of the standard lattice in $Y_{k_v}\oplus L_{k_v}$ (notation of Section 2), and $\tilde{w}^0_{(l_0,l_0)/2}$ is the normalized unramified function in $\tilde{W}_{\psi_v}(\sigma_v,\frac{1}{2}(l_0,l_0))$, then

$$w_{\zeta^0_v}(h)=\int_{U_v\backslash\mathrm{SL}(2,k_v)}\omega_{\psi_v}(g,h)\phi^0_v(\varepsilon_+;l_0)\tilde{w}^0_{(l_0,l_0)/2}(g)dg.$$

By the Iwasawa decomposition (and the formulas for ω_{ψ_v}),

$$w_{\zeta_v^0}(h)=\int_{k_v^*}\omega_{\psi_v}\left(\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix},h\right)\phi_v^0(\varepsilon_+;l_0)\tilde{w}^0_{(l_0,l_0)/2}\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}|t|^2d^*t$$

$$=\int_{k_v^*}\chi_{\psi_v}(t^{-m+2})((-1)^{[m/2]}d_v,t)_v|t|^{-(m-2)/2}\omega_{\psi_v}(1,h)\phi_v^0(t\varepsilon_+;t^{-1}l_0)$$

$$\times\tilde{w}^0_{(l_0,l_0)/2}\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}|t|^2d^*t.$$

Here d_v is the discriminant of the quadratic form on X_{k_v}, $(\ ,\ )_v$ is the Hilbert symbol on k_v^* and χ_{ψ_v} is the character of $\tilde{k}_v^*$ appearing in the formulas for ω_{ψ_v}. In particular

$$w_{\zeta_v^0}\begin{pmatrix} x & & \\ & I & \\ & & x^{-1}\end{pmatrix}$$

$$=|x|\int_{k_v^*}\chi_{\psi_v}(t^{-m})\chi_{d_v}(t)|t|^{(6-m)/2}\phi_v^0(tx\varepsilon_+;t^{-1}l_0)\tilde{w}^0_{(l_0,l_0)/2}\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}d^*t.$$

We abbreviated $((-1)^{[m/2]}d_v,t)_v=\chi_{d_v}(t)$. Write $\phi_v^0(a\varepsilon_+;bl_0)=\tilde{\phi}_v^0(a,b)$ where $\tilde{\phi}_v^0$ is the characteristic function of the standard lattice in k_v^2. Thus (for Re(s) large)

$$I_v^0(s)=L_v\left(1,s+\frac{m-3}{2}\right)$$

$$\cdot\int_{k_v^*\times k_v^*}\chi_{\psi_v}(t^{-m})\chi_{d_v}(t)\tilde{\phi}_v^0(tx,t^{-1})$$

$$\cdot\tilde{w}^0_{(l_0,l_0)/2}\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}|t|^{(6-m)/2}|x|^{s-(m-5)/2}d^*t\,d^*x.$$

Changing t to t^{-1} and noting that

$$\tilde{w}^0_{(l_0,l_0)/2}\begin{pmatrix} t^{-1} & 0\\ 0 & t\end{pmatrix}=w^0_{-(l_0,l_0)/2}\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix},$$

where $w^0_{-(l_0,l_0)/2}$ is the normalized unramified function in the Whittaker model of σ_v with respect to $\left(\begin{smallmatrix} 1 & *\\ 0 & 1\end{smallmatrix}\right)$ and the character $\psi^{-(l_0,l_0)/2}$, we get (for Re(s) large)

$$I_v^0(s) = L_v\left(1, s+\frac{m-3}{2}\right)\int_{|x|\le|t|\le 1}\chi_{\psi_v}(t^{-m})\chi_{d_v}(t)|t|^{(m-6)/2}$$

$$\times\, w^0_{-(l_0,l_0)/2}\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}|x|^{s-(m-5)/2}d^*(t,x)$$

$$= L_v\left(1, s+\frac{m-3}{2}\right)L_v\left(1, s-\frac{m-5}{2}\right)\int_{|t|\le 1}\chi_{\psi_v}(t^{-m})\chi_{d_v}(t)$$

$$\times\, w^0_{-(l_0,l_0)/2}\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}|t|^{s-1/2}d^*t.$$

Denote the last integral by $J_v^0(s)$. We compute it explicitly.

Case I (m even). In this case let w_0 be the normalized unramified Whittaker function on $GL(2, k_v)$, extending $w^0_{-(l_0,l_0)/2}$ for the representation $\pi'_v = \mathrm{Ind}_{B_v}^{GL(2,k_v)}\beta_v\otimes\beta_v^{-1}$, where B_v is the standard Borel subgroup of $GL(2, k_v)$ and β_v is an unramified character of k_v^*, such that $\sigma_v = \mathrm{Ind}_{B'_v}^{SL(2,k_v)}\mu_v$, where $B'_v = B_v \cap SL(2, k_v)$ and $\mu_v = \beta_v^2$. Then we have

$$J_v^0(s) = \int_{|t|\le 1} w_0\begin{pmatrix} t^2 & 0\\ 0 & 1\end{pmatrix}|t|^{(s+1/2)-1}\chi_{d_v}(t)d^*t.$$

This integral is computed in [G.J],

$$J_v^0(s) = \frac{L((\pi'_v\otimes\chi_{d_v})\times\hat{\pi}'_v, s+\frac{1}{2})}{L_v(1, 2s+1)L(\chi_{d_v}, s+\frac{1}{2})}$$

$$= \frac{L(\chi_{d_v}, s+\frac{1}{2})L(\mu_v\chi_{d_v}, s+\frac{1}{2})L(\mu_v^{-1}\chi_{d_v}, s+\frac{1}{2})}{L_v(1, 2s+1)}.$$

Note that since σ_v is unitary, generic and unramified, then $|\mu_v(t)| = |t|^{a_v}$ with $|a_v| < 1$.

Case II (m odd). In this case

$$J_v^0(s) = \int_{|t|\le 1}\chi_{\psi_v}(t)\chi_{d_v}(t)|t|^{s-1/2}w^0_{-(l_0,l_0)/2}\begin{pmatrix} t & 0\\ 0 & t^{-1}\end{pmatrix}d^*t.$$

Assume that σ_v corresponds to the unramified character $\tilde{\mu}_v$ of $\tilde{k}_v^*$, so that σ_v is contained in $\mathrm{Ind}_{\tilde{B}_v}^{\widetilde{SL}(2,K_v)}\tilde{\mu}_v$. If σ_v is an even Weil representation (corresponding to $\psi_v^{-(l_0,l_0)}$) then

$$J_v^0(s) = \int_{|t|\le 1}\chi_{d_v(l_0,l_0)}(t)|t|^s d^*t = L_v(\chi_{d_v(l_0,l_0)}, s).$$

Otherwise, σ_v lifts to an infinite dimensional representation of PGL(2, k_v), via the theta-correspondence with respect to $\psi_v^{-(l_0,l_0)}$. Denote the resulting representation of PGL(2, k_v) by τ_v, then from Proposition 2.1 in [G.PS] we conclude that

$$J_v^0(s) = \frac{L(\tau_v \otimes \chi_{d_v(l_0,l_0)}, s + \frac{1}{2})}{L(\chi_{d_v(l_0,l_0)}, s + 1)} .$$

In this case τ_v is of the form $\mathrm{Ind}_{B_v}^{\mathrm{GL}(2,k_v)} \mu_v \otimes \mu_v^{-1}$, where μ_v is an unramified character of k_v^*, and our assumptions on σ_v imply that $|\mu_v(t)| = |t|^{a_v}$, with $|a_v| < \frac{1}{2}$. In this case

$$J_v^0(s) = \frac{L(\mu_v \chi_{d_v(l_0,l_0)}, s + \frac{1}{2}) L(\mu_v^{-1} \chi_{d_v(l_0,l_0)}, s + \frac{1}{2})}{L(\chi_{d_v(l_0,l_0)}, s + 1)} .$$

Finally, we have

$$I_{l_0}(\varphi, f_s) = I_\infty(s) L^\Omega\left(1, s + \frac{m+3}{2}\right) L^\Omega\left(1, s - \frac{m-5}{2}\right) F(\sigma, s)$$

where

$$F(\sigma, s) = \prod_{v \notin \Omega} J_v^0(s).$$

By our estimates on the exponents a_v, we see that $F(\sigma, s)$ converges absolutely for $\mathrm{Re}(s) \geqq \frac{3}{2}$ when m is even, and for $\mathrm{Re}(s) > 1$ when m is odd. We see that $s = (m-3)/2$ lies in this domain for $m \geqq 6$. This proves the lemma. □

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